Hierarchy of fluids models for magnetized and collisional plasmas.

Part I: Bi-fluid and Full-MHD models

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In this report we propose to study the different fluids models used to describe collisional plasmas. The main objective is to derive the different models and give for all these models an energy balance law. Firstly we will derive the two fluid models using the moments of the two species Vlasov-Maxwell equations. After this, using the quasi neutrality assumption which comes from that the characteristic velocity of the plasma is very small comparing the light velocity, we will derive the Extended MHD model. This model is the main model of this document. Firstly we will give an energy estimate and after we propose an ordering of this equation which allows to obtain a family of MHD models. Lot of computation proposed in the document can be found in the PhD thesis [MARTIN 2013].

1 Derivation of two fluid mode using Vlasov equations

The first part of this report aims at deriving the two fluids system (with closures) using the firsts second and third moments of the Vlasov-Maxwell equations. The Vlasov equation admit an energy estimate and we will keep this estimate for the fluid model.

1.1 Vlasov Equation

We consider that our plasmas can be described by the following two species Vlasov-Maxwell equations. We consider the distribution function associated with the species s noted $f_s(t, \boldsymbol{x}, \boldsymbol{v})$. The spatial variable is $\boldsymbol{x} \in D_{\boldsymbol{x}}$ and velocity variable is $\boldsymbol{v} \in \mathbb{R}^3$. The charge of particle is called q_s. The 2 species Vlasov - Maxwell equation is given by

Two species Vlasov-Maxwell equations

$$\begin{cases} \partial_t f_s + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}}(f_s) + \dfrac{q_s}{m_s} (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \nabla_{\boldsymbol{v}} f_s = C_s \\[2ex] \dfrac{1}{c^2} \partial_t \boldsymbol{E} - \nabla \times \boldsymbol{B} = -\mu_0 \boldsymbol{J} \\[2ex] \partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E} \\[2ex] \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \cdot \boldsymbol{E} = \dfrac{\sigma}{\varepsilon_0} \end{cases} \tag{1.1}$$

with $\sigma = \sum_s \sigma_s = \sum_s q_s n_s$. The collision operator C_s is given by $C_s = \sum_{s'} C_{ss'}$. Since the number of particles, the moment and the total energy for one specie or for the sum of the two species are not changed by the collisions at the point $\boldsymbol{x}$ we have

- $\displaystyle\int_{\mathbb{R}^3} C \ \mathrm{d}v_{ss'} d\boldsymbol{v} = 0$
- $\displaystyle\int_{\mathbb{R}^3} m \ \mathrm{d}v_s \boldsymbol{v} C_{ss} d\boldsymbol{v} = 0$
- $\displaystyle\int_{\mathbb{R}^3} m \ \mathrm{d}v_s |\boldsymbol{v}|^2 C_{ss} d\boldsymbol{v} = 0$
- $\displaystyle\int_{\mathbb{R}^3} m \ \mathrm{d}v_s \boldsymbol{v} C_{ss'} d\boldsymbol{v} + \int_{\mathbb{R}^3} m \ \mathrm{d}v_{s'} \boldsymbol{v} C_{s's} d\boldsymbol{v} = 0$

- $\int_{\mathbb{R}^3} m \ \mathrm{d}v_s |\boldsymbol{v}|^2 C_{ss'} d\boldsymbol{v} + \int_{\mathbb{R}^3} m \ \mathrm{d}v_{s'} |\boldsymbol{v}|^2 C_{s's} d\boldsymbol{v} = 0$

For now, we do not consider the case where we have a external source. It will however be important to add external source term when modeling realistic tokamak physics.

Lemma

The equations (1.1) satisfy

$$\frac{d}{dt}\left(\frac{1}{2}\sum_s \int_\Omega\int_{\mathbb{R}^3} m_s f_s \mid \boldsymbol{v} \mid^2 \ \mathrm{d}v\mathrm{d}x + \frac{1}{2\mu_0 c^2}\int_\Omega \mid \boldsymbol{E} \mid^2 \ \mathrm{d}x + \frac{1}{2\mu_0}\int_\Omega \mid \boldsymbol{B} \mid^2 \ \mathrm{d}x\right) = 0.$$

Proof. We multiply the kinetic equation by $\frac{|\boldsymbol{v}|^2}{2}$, we sum the two equations on the two species and we integrate in velocity and space. We obtain

$$\sum_s \left[\partial_t \int_\Omega\int_{\mathbb{R}^3}\left(m_s\frac{\mid \boldsymbol{v} \mid^2}{2} f_s\right) \ \mathrm{d}v\mathrm{d}x + \int_\Omega\int_{\mathbb{R}^3}\left(m_s\frac{\mid \boldsymbol{v} \mid^2}{2}\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f_s\right) \ \mathrm{d}v\mathrm{d}x + \int_\Omega\int_{\mathbb{R}^3}\left(q_s\left(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}\right)\cdot\nabla_{\boldsymbol{v}} f_s\right) \ \mathrm{d}v\mathrm{d}x\right] = 0,$$

$$\sum_s \left[\partial_t \int_\Omega\int_{\mathbb{R}^3}\left(m_s\frac{\mid \boldsymbol{v} \mid^2}{2} f_s\right) \ \mathrm{d}v\mathrm{d}x + \int_\Omega\int_{\mathbb{R}^3} m_s\nabla_{\boldsymbol{x}}\cdot\left(\frac{\mid \boldsymbol{v} \mid^2}{2} f_s\right) \ \mathrm{d}v\mathrm{d}x + \int_\Omega\int_{\mathbb{R}^3}\left(q_s\left(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}\right)\cdot\nabla_{\boldsymbol{v}} f_s\right) \ \mathrm{d}v\mathrm{d}x\right] = 0.$$

We have obtained this using the previous assumptions on the collision operators. Using the free divergence theorem we obtain

$$\sum_s \partial_t \int_\Omega\int_{\mathbb{R}^3}\left(m_s\frac{\mid \boldsymbol{v} \mid^2}{2} f_s\right) \ \mathrm{d}v\mathrm{d}x + \sum_s \int_\Omega\int_{\mathbb{R}^3}\left(\frac{\mid \boldsymbol{v} \mid^2}{2}\left(q_s\left(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}\right)\cdot\nabla_{\boldsymbol{v}} f_s \ \mathrm{d}v\mathrm{d}x\right)\right) = 0.$$

Integrating by part in the velocity space we obtain

$$\sum_s \partial_t \int_\Omega\int_{\mathbb{R}^3}\left(m_s\frac{\mid \boldsymbol{v} \mid^2}{2} f_s\right) \ \mathrm{d}v\mathrm{d}x - \sum_s \int_\Omega\int_{\mathbb{R}^3}\left(f_s\left(q_s\left(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}\right)\cdot\nabla_{\boldsymbol{v}}\left(\frac{\mid \boldsymbol{v} \mid^2}{2}\right)\right)\right) \ \mathrm{d}v\mathrm{d}x = 0,$$

$$\sum_s \partial_t \int_\Omega\int_{\mathbb{R}^3}\left(m_s\frac{\mid \boldsymbol{v} \mid^2}{2} f_s\right) \ \mathrm{d}v\mathrm{d}x - \sum_s \int_\Omega\int_{\mathbb{R}^3}\left(f_s\left(q_s\left(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}\right)\cdot\boldsymbol{v}\right)\right) \ \mathrm{d}v\mathrm{d}x = 0,$$

$$\sum_s \partial_t \int_\Omega\int_{\mathbb{R}^3}\left(m_s\frac{\mid \boldsymbol{v} \mid^2}{2} f_s\right) \ \mathrm{d}v\mathrm{d}x - \sum_s \int_\Omega q_s\boldsymbol{E}\cdot\left(\int_{\mathbb{R}^3}\boldsymbol{v} \ \mathrm{d}v f_s\right) - \int_\Omega\int_{\mathbb{R}^3} \ \mathrm{d}v\mathrm{d}x q_s\left(\boldsymbol{v}\times\boldsymbol{B}\right)\cdot\boldsymbol{v} f_s \ \mathrm{d}x = 0.$$

We have $(\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{v} = 0$. Now we integrate in the velocity space the Vlasov-Maxwell equation and we define $\rho = \int_{\mathbb{R}^3} f \ \mathrm{d}v = \sum_s m_s \int_{\mathbb{R}^3} f_s \ \mathrm{d}v$ and $\boldsymbol{J} = \int_{\mathbb{R}^3}\boldsymbol{v} f \ \mathrm{d}v = \sum_s q_s \int_{\mathbb{R}^3}\boldsymbol{v} f_s \ \mathrm{d}v$. Consequently we obtain that

$$\partial_t \int_\Omega\int_{\mathbb{R}^3}\left(m_s\frac{\mid \boldsymbol{v} \mid^2}{2} f_s\right) \ \mathrm{d}v\mathrm{d}x - \int_\Omega \boldsymbol{E}\cdot\boldsymbol{J} \ \mathrm{d}x = 0.$$

Now we multiply the Ampere equation (equation which describe the evolution of the electric field in the Maxwell equation) by the electric field and we integrate to obtain

$$\frac{1}{2c^2}\partial_t \int_\Omega \mid \boldsymbol{E} \mid^2 \ \mathrm{d}x - \int_\Omega (\nabla\times\boldsymbol{B})\cdot\boldsymbol{E} \ \mathrm{d}x = -\int_\Omega \mu_0\boldsymbol{J}\cdot\boldsymbol{E} \ \mathrm{d}x.$$

Consequently we obtain

$$\partial_t \sum_s \int_\Omega \int_{\mathbb{R}^3} \left(\frac{m_s \mid \boldsymbol{v} \mid^2}{2} f_s \right) \, \mathrm{d}v \mathrm{d}x + \frac{1}{2\mu_0 c^2} \partial_t \int_\Omega \mid \boldsymbol{E} \mid^2 \, \mathrm{d}x - \frac{1}{\mu_0} \int_\Omega (\nabla \times \boldsymbol{B}) \cdot \boldsymbol{E} \, \mathrm{d}x = 0. \qquad (1.2)$$

Now we consider the Faraday law (equation which describe the evolution of the Magnetic field). We multiply by the magnetic field and integrate.

$$\int_\Omega \partial_t \boldsymbol{B} \cdot \boldsymbol{B} \, \mathrm{d}x + \int_\Omega (\nabla \times \boldsymbol{E}) \cdot \boldsymbol{B} \, \mathrm{d}x = 0$$
$$\frac{1}{2} \partial_t \int_\Omega \mid \boldsymbol{B} \mid^2 \, \mathrm{d}x + \int_\Omega (\nabla \times \boldsymbol{B}) \cdot \boldsymbol{E} \, \mathrm{d}x = 0$$

Plugging the last relation in the equation (1.2) we obtain the results. ■

1.2 Derivation of the two fluids model

To derive the two fluid models, we take the moments of the kinetic equation where the distribution function is close to the Maxwellian distribution defined by

$$f_M = n \left(\frac{m}{2\pi T} \right)^{\frac{3}{2}} e^{-\frac{m \mid \boldsymbol{v} - \boldsymbol{u} \mid^2}{2T}} = n \frac{e^{-\frac{\mid \boldsymbol{v} - \boldsymbol{u} \mid^2}{v_T^2}}}{\pi^{\frac{3}{2}} v_T^3}$$

with $\boldsymbol{u} = \frac{1}{n} \int_{\mathbb{R}^3} \boldsymbol{v} f_M \, \mathrm{d}v$ and the thermal velocity $v_T = \sqrt{\frac{2T}{m}}$. The distribution function is close to the Maxwellian when the dynamic of the plasma is dominated by the collisions (Coulomb collisions). For this we introduce the function $g(\boldsymbol{v})$. The method consist to multiply the kinetic equation by $g(\boldsymbol{v})$, integrate on the velocity space for $g(\boldsymbol{v}) = 1, \quad m_s \boldsymbol{v}, \quad m_s \mid \boldsymbol{v} \mid^2$. For the following we assume that the distribution function is a compact function in the velocity space. Consequently the distribution tends to zero at the boundary of the velocity space. Here we neglected the source.

1.3 Continuity equations

To obtains these equations we use $g(\mathbf{w}) = 1$. We obtain

$$\partial_t \int_{\mathbb{R}^3} f_s \, \mathrm{d}v + \int_{\mathbb{R}^3} \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} (f_s) \, \mathrm{d}v + \int_{\mathbb{R}^3} \frac{q_s}{m_s} (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \nabla_{\boldsymbol{v}} f_s \, \mathrm{d}v = \int_{\mathbb{R}^3} C_s \, \mathrm{d}v \qquad (1.3)$$

and we define

$$n_s = \int_{\mathbb{R}^3} f_s \, \mathrm{d}v, \quad \boldsymbol{u}_s = \frac{1}{n_s} \int_{\mathbb{R}^3} f_s \, \mathrm{d}v$$

By definition since $\boldsymbol{v}$ does not depend of $\boldsymbol{x}$ and since we integrate on the velocity space we can write

$$\int_{\mathbb{R}^3} \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} (f_s) \, \mathrm{d}v = \int_\Omega \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{v} f_s) \, \mathrm{d}x = \nabla_{\boldsymbol{x}} \cdot \int_{\mathbb{R}^3} \boldsymbol{v} f_s \, \mathrm{d}v$$

Consequently we obtain

$$\partial_t n_s + \nabla \cdot (n_s \boldsymbol{u}_s) + \int_{\mathbb{R}^3} \frac{q_s}{m_s} (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \nabla_{\boldsymbol{v}} f_s \, \mathrm{d}v = \int_{\mathbb{R}^3} C_s \, \mathrm{d}v$$

Using the properties of the collision operator we remark that the integral on the velocity space of C_s is equal to zero. Now we remark that

$$(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\nabla_{\boldsymbol{v}} f_s = \nabla_{\boldsymbol{v}}\cdot\left((\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\, f_s\right)$$

since $\nabla_{\boldsymbol{v}}\cdot(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}) = (\nabla_{\boldsymbol{v}}\times\boldsymbol{v})\cdot\boldsymbol{B} - \boldsymbol{v}\cdot(\nabla_{\boldsymbol{v}}\times\boldsymbol{B}) = 0$. Using the free-divergence theorem we obtain that

$$\int_{\mathbb{R}^3}\nabla_{\boldsymbol{v}}\cdot\left((\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\, f_s\right)\,\mathrm{d}v = \int_{\mathbb{R}^3}\left((\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\, f_s\right)\,\mathrm{d}v = 0$$

since the distribution is a support compact function in the velocity space. At the end we have obtain

$$\partial_t n_s + \nabla\cdot(n_s\boldsymbol{u}_s) = 0$$

Multiplying the equations by m_s we obtain

$$\boxed{\partial_t\rho_s + \nabla\cdot(\rho_s\boldsymbol{u}_s) = 0}$$

with $\rho_s = m_s n_s$.

1.4 Momentum equations

To obtains these equations we use $g(\mathbf{w}) = m_s\boldsymbol{v}$. We obtain

$$\int_{\mathbb{R}^3} m_s\boldsymbol{v}\partial_t f_s\,\mathrm{d}v + \int_{\mathbb{R}^3} m_s\boldsymbol{v}\nabla_{\boldsymbol{x}}\cdot(\boldsymbol{v}f_s)\,\mathrm{d}v + \int_{\mathbb{R}^3} m_s\boldsymbol{v}\nabla_{\boldsymbol{v}}\cdot(\frac{q_s}{m_s}\left(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}\right) f_s)\,\mathrm{d}v$$
$$= \int_{\mathbb{R}^3} m_s\boldsymbol{v}C_s\,\mathrm{d}v$$

and we define $\boldsymbol{k} = \boldsymbol{v} - \boldsymbol{u}_s$ and

$$p_s = \int_{\mathbb{R}^3} m_s\frac{1}{3}|\boldsymbol{k}|^2 f_s\,\mathrm{d}v = n_s T_s, \quad \overline{\overline{\boldsymbol{\Pi}}}_s = \int_{\mathbb{R}^3} m_s(\boldsymbol{k}\otimes\boldsymbol{k} - \frac{1}{3}|\boldsymbol{k}|^2 I_d) f_s\,\mathrm{d}v$$

and

$$\mathbf{R}_s = \int_{\mathbb{R}^3} m_s\boldsymbol{k}\sum_{s'} C_{ss'}\,\mathrm{d}v$$

The coefficient p_s is the isotropic pressure, $\overline{\overline{\boldsymbol{\Pi}}}_s$ the stress tensor and $\mathbf{R}_s$ the friction coefficient. It is simple to obtain that $\int_{\mathbb{R}^3} m_s\boldsymbol{v}\partial_t f_s\,\mathrm{d}v = \partial_t(\rho_s\boldsymbol{u}_s)$. Now we consider the second term

$$\begin{aligned}\int_{\mathbb{R}^3} m_s\boldsymbol{v}\nabla_{\boldsymbol{x}}\cdot(\boldsymbol{v}f_s)\,\mathrm{d}v &= \int_{\mathbb{R}^3}\nabla_{\boldsymbol{x}}\cdot(m_s\boldsymbol{v}\otimes\boldsymbol{v}f_s)\,\mathrm{d}v\\ &= \nabla_{\boldsymbol{x}}\cdot\int_{\mathbb{R}^3}(m_s\boldsymbol{v}\otimes\boldsymbol{v}f_s)\,\mathrm{d}v\\ &= \nabla_{\boldsymbol{x}}\cdot\int_{\mathbb{R}^3}\left(m_s\boldsymbol{u}_s\otimes\boldsymbol{u}_s f_s + m_s(\boldsymbol{u}_s\otimes\boldsymbol{k}+\boldsymbol{k}\otimes\boldsymbol{u}_s)f_s + m_s\boldsymbol{k}\otimes\boldsymbol{k}f_s\right)\,\mathrm{d}v\end{aligned}$$

By definition of $\boldsymbol{k}$ we obtain that $\int_{\mathbb{R}^3} \boldsymbol{k} f_s \, \mathrm{d}v = 0$ (because $\boldsymbol{u}_s$ doest not depend of $\boldsymbol{v}$), therefore

$$\begin{aligned}
\int_{\mathbb{R}^3} m_s \boldsymbol{v} \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{v} f_s) \, \mathrm{d}v &= \nabla_{\boldsymbol{x}} \cdot \int_{\mathbb{R}^3} \left(m_s \boldsymbol{u}_s \otimes \boldsymbol{u}_s f_s + m_s \boldsymbol{k} \otimes \boldsymbol{k} f_s \right) \, \mathrm{d}v \\
&= \nabla_{\boldsymbol{x}} \cdot (\rho_s \boldsymbol{u}_s \otimes \boldsymbol{u}_s) + \nabla_{\boldsymbol{x}} \cdot \left(\int_{\mathbb{R}^3} m_s \boldsymbol{k} \otimes \boldsymbol{k} f_s \, \mathrm{d}v \right) \\
&= \nabla_{\boldsymbol{x}} \cdot (\rho_s \boldsymbol{u}_s \otimes \boldsymbol{u}_s) + \nabla_{\boldsymbol{x}} p_s + \nabla_{\boldsymbol{x}} \cdot \overline{\overline{\overline{\mathbf{\Pi}}}}_s
\end{aligned}$$

By definition of the stress tensor and the isotropic pressure. After this we study the term linked to the electric and magnetic field.

$$\begin{aligned}
\int_{\mathbb{R}^3} m_s \boldsymbol{v} \nabla_{\boldsymbol{v}} \cdot \left(\frac{q_s}{m_s} \left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} \right) f_s\right) \mathrm{d}v &= -\int_{\mathbb{R}^3} m_s \nabla_{\boldsymbol{v}}(\boldsymbol{v}) \cdot \left(\frac{q_s}{m_s} \left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} \right) f_s\right) \mathrm{d}v \\
&= -\int_{\mathbb{R}^3} \left(q_s \left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} \right) f_s\right) \mathrm{d}v \\
&= -q_s n_s (\boldsymbol{E} + \boldsymbol{u}_s \times \boldsymbol{B})
\end{aligned}$$

To obtain the previous result we use an integration by part and $\nabla_{\boldsymbol{v}} \boldsymbol{v} = 1 I_d$. To finish we study the collision term

$$\begin{aligned}
\int_{\mathbb{R}^3} m_s \boldsymbol{v} C_s \, \mathrm{d}v &= \int_{\mathbb{R}^3} m_s \boldsymbol{v} \sum_{s'} C_{ss'} \, \mathrm{d}v \\
&= \int_{\mathbb{R}^3} m_s \boldsymbol{k} \sum_{s'} C_{ss'} \, \mathrm{d}v + \int_{\mathbb{R}^3} m_s \boldsymbol{u}_s \sum_{s'} C_{ss'} \, \mathrm{d}v \\
&= \int_{\mathbb{R}^3} m_s \boldsymbol{k} \sum_{s'} C_{ss'} \, \mathrm{d}v + m_s \boldsymbol{u}_s \int_{\mathbb{R}^3} \sum_{s'} C_{ss'} \, \mathrm{d}v \\
&= \mathbf{R}_s
\end{aligned}$$

At the end we have obtain

$$\boxed{\partial_t (\rho_s \boldsymbol{u}_s) + \nabla_{\boldsymbol{x}} \cdot (\rho_s \boldsymbol{u}_s \otimes \boldsymbol{u}_s) + \nabla_{\boldsymbol{x}} p_s + \nabla_{\boldsymbol{x}} \cdot \overline{\overline{\overline{\mathbf{\Pi}}}}_s = \sigma_s \boldsymbol{E} + \boldsymbol{J}_s \times \boldsymbol{B} + \mathbf{R}_s}$$

since $\boldsymbol{J}_s = \sigma_s \boldsymbol{u}_s$.

1.5 Energy equations

To obtain these equations we use $g(\mathbf{w}) = m_s \frac{|\boldsymbol{v}|^2}{2}$. This leads

$$\begin{aligned}
&\int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{v}|^2}{2} \partial_t f_s \, \mathrm{d}v + \int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{v}|^2}{2} \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{v} f_s) \, \mathrm{d}v + \int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{v}|^2}{2} \nabla_{\boldsymbol{v}} \cdot \left(\frac{q_s}{m_s} \left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} \right) f_s\right) \mathrm{d}v \\
&= \int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{v}|^2}{2} C_s \, \mathrm{d}v.
\end{aligned}$$

We consider the first term

$$
\begin{aligned}
\int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{v}|^2}{2} \partial_t f_s \; \mathrm{d}v =& \partial_t \int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{v}|^2}{2} f_s \; \mathrm{d}v \\
=& \partial_t \left(\int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{v} - \boldsymbol{u}_s + \boldsymbol{u}_s|^2}{2} f_s \; \mathrm{d}v \right) \\
=& \partial_t \left(\int_{\mathbb{R}^3} (m_s \frac{|\boldsymbol{k}|^2}{2} f_s + m_s \boldsymbol{k} \boldsymbol{u}_s f_s + m_s \frac{|\boldsymbol{u}_s|^2}{2} f_s) \; \mathrm{d}v \right) \\
=& \partial_t \left(\int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{k}|^2}{2} f_s \; \mathrm{d}v \right) + \partial_t \left(\boldsymbol{u}_s \int_{\mathbb{R}^3} m_s \boldsymbol{k} f_s \; \mathrm{d}v \right) + \partial_t \left(\int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{u}_s|^2}{2} f_s \; \mathrm{d}v \right) \\
=& \partial_t \frac{3}{2} \left(\int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{k}|^2}{3} f_s \; \mathrm{d}v \right) + \partial_t \left(\frac{|\boldsymbol{u}_s|^2}{2} \int_{\mathbb{R}^3} m_s f_s \; \mathrm{d}v \right) \\
=& \partial_t \left(\frac{3}{2} p_s + \frac{1}{2} \rho_s |\boldsymbol{u}_s|^2 \right) = \partial_t (\rho_s \epsilon_s),
\end{aligned}
$$

which comes from the fact that $\boldsymbol{u}_s$ does not depend on $\boldsymbol{v}$ and $\int_{\mathbb{R}^3} \boldsymbol{k} f_s \; \mathrm{d}v = 0$. We then define the heat flux $\mathbf{q}$ as

$$
\mathbf{q} = \int_{\mathbb{R}^3} m_s \boldsymbol{k} \frac{|\boldsymbol{k}|^2}{2} f_s \; \mathrm{d}v.
$$

Considering the second term we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^3} \frac{|\boldsymbol{v}|^2}{2} \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{v} f_s) \; \mathrm{d}v =& \nabla_{\boldsymbol{x}} \cdot \left(\int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{v}|^2}{2} \boldsymbol{v} f_s \; \mathrm{d}v \right) \\
=& \nabla_{\boldsymbol{x}} \cdot \left(\int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{v}|^2}{2} \boldsymbol{k} f_s \; \mathrm{d}v \right) + \nabla_{\boldsymbol{x}} \cdot \left(\int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{v}|^2}{2} \boldsymbol{u}_s f_s \; \mathrm{d}v \right) \\
=& \nabla_{\boldsymbol{x}} \cdot \left(\int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{v}|^2}{2} \boldsymbol{k} f_s \; \mathrm{d}v \right) + \nabla_{\boldsymbol{x}} \cdot \left(\boldsymbol{u}_s \int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{v}|^2}{2} f_s \; \mathrm{d}v \right) \\
=& \nabla_{\boldsymbol{x}} \cdot \left(\int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{v}|^2}{2} \boldsymbol{k} f_s \; \mathrm{d}v \right) + \nabla_{\boldsymbol{x}} \cdot \left(\rho_s \boldsymbol{u}_s \epsilon_s \right).
\end{aligned}
$$

The last result uses the previous computations. Now considering $\nabla_{\boldsymbol{x}} \cdot \left(\int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{v}|^2}{2} \boldsymbol{k} f_s \; \mathrm{d}v \right)$, we have

$$
\begin{aligned}
\nabla_{\boldsymbol{x}} \cdot \left(\int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{v}|^2}{2} \boldsymbol{k} f_s \; \mathrm{d}v \right) =& \nabla_{\boldsymbol{x}} \cdot \left(\int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{k}|^2}{2} \boldsymbol{k} f_s \; \mathrm{d}v + \int_{\mathbb{R}^3} m_s (\boldsymbol{k} \boldsymbol{u}_s) \boldsymbol{k} f_s \; \mathrm{d}v + \int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{u}_s|^2}{2} \boldsymbol{k} f_s \; \mathrm{d}v \right) \\
=& \nabla_{\boldsymbol{x}} \cdot \left(\int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{k}|^2}{2} \boldsymbol{k} f_s \; \mathrm{d}v + \int_{\mathbb{R}^3} m_s (\boldsymbol{k} \boldsymbol{u}_s) \boldsymbol{k} f_s \; \mathrm{d}v + \frac{|\boldsymbol{u}_s|^2}{2} \int_{\mathbb{R}^3} m_s \boldsymbol{k} f_s \; \mathrm{d}v \right) \\
=& \nabla_{\boldsymbol{x}} \cdot (\mathbf{q}) + \nabla_{\boldsymbol{x}} \cdot \left(\int_{\mathbb{R}^3} m_s (\boldsymbol{k} \boldsymbol{u}_s) \boldsymbol{k} f_s \; \mathrm{d}v \right) \\
=& \nabla_{\boldsymbol{x}} \cdot (\mathbf{q}) + \nabla_{\boldsymbol{x}} \cdot \left(\overline{\overline{\boldsymbol{\Pi}}} \cdot \boldsymbol{u}_s \right) + \nabla_{\boldsymbol{x}} \cdot \left(p_s \boldsymbol{u}_s \right),
\end{aligned}
$$

therefore

$$
\int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{v}|^2}{2} \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{v} f_s) \; \mathrm{d}v = \nabla_{\boldsymbol{x}} \cdot (\rho_s \boldsymbol{u}_s \epsilon_s) + \nabla_{\boldsymbol{x}} \cdot \left(\overline{\overline{\boldsymbol{\Pi}}} \cdot \boldsymbol{u}_s \right) + \nabla_{\boldsymbol{x}} \cdot (p_s \boldsymbol{u}_s)
$$

We continue by studying the term linked to the magnetic and electrical field, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{v}|^2}{2} \nabla_{\boldsymbol{v}} \cdot \left(\frac{q_s}{m_s} \left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right) f_s\right) \mathrm{d}v &= -\int_{\mathbb{R}^3} q_s \left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right) f_s \cdot \nabla_{\boldsymbol{v}} \left(\frac{|\boldsymbol{v}|^2}{2}\right) \mathrm{d}v \\
&= -\int_{\mathbb{R}^3} q_s \left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right) f_s \cdot \boldsymbol{v} \, \mathrm{d}v \\
&= -\sigma_s \boldsymbol{E} \cdot \boldsymbol{u}_s - \int_{\mathbb{R}^3} q_s (\boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v} f_s \, \mathrm{d}v \\
&= -\sigma_s \boldsymbol{E} \cdot \boldsymbol{u}_s,
\end{aligned}$$

having used that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (\mathbf{a} \times \mathbf{a}) \cdot \mathbf{b} = 0$. To find the terms associated with the collision we define

$$Q_{\Delta_s} = \int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{k}|^2}{2} \sum_{s'} C_{ss'} \, \mathrm{d}v.$$

We finish by the collision term.

$$\begin{aligned}
\int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{v}|^2}{2} C_s \, \mathrm{d}v &= \int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{k}|^2}{2} C_s \, \mathrm{d}v + \int_{\mathbb{R}^3} m_s \boldsymbol{u}_s \boldsymbol{k} C_s \, \mathrm{d}v + \int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{u}_s|^2}{2} C_s \, \mathrm{d}v \\
&= \int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{k}|^2}{2} \sum_{s'} C_{ss'} \, \mathrm{d}v + \int_{\mathbb{R}^3} m_s \boldsymbol{u}_s \boldsymbol{k} \sum_{s'} C_{ss'} \, \mathrm{d}v + \int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{u}_s|^2}{2} \sum_{s'} C_{ss'} \, \mathrm{d}v \\
&= Q_{\Delta_s} + \mathbf{R}_s \cdot \boldsymbol{u}_s.
\end{aligned}$$

The last term is equal to zero by definition of operator collision. We obtain at the end

$$\boxed{\partial_t(\rho_s \epsilon_s) + \nabla_{\boldsymbol{x}} \cdot (\rho_s \boldsymbol{u}_s \epsilon_s) + \nabla_{\boldsymbol{x}} \cdot \left(\overline{\overline{\boldsymbol{\Pi}}} \cdot \boldsymbol{u}_s\right) + \nabla_{\boldsymbol{x}} \cdot (p_s \boldsymbol{u}_s) + \nabla_{\boldsymbol{x}} \cdot (\mathbf{q}) = \sigma_s \boldsymbol{E} \cdot \boldsymbol{u}_s + Q_{\Delta_s} + \mathbf{R}_s \cdot \boldsymbol{u}_s}$$

1.6 Final Bi-fluid model and Energy

The final bi-fluid model is given by

Bi-fluid model

$$\begin{cases} \partial_t \rho_i + \nabla \cdot (\rho_i \boldsymbol{u}_i) = 0 \\ \partial_t \rho_e + \nabla \cdot (\rho_e \boldsymbol{u}_e) = 0 \\ \\ \partial_t (\rho_i \boldsymbol{u}_i) + \nabla \cdot (\rho_i \boldsymbol{u}_i \otimes \boldsymbol{u}_i) + \nabla p_i = \sigma_i \mathbf{E} + \boldsymbol{J}_i \times \boldsymbol{B} - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_i + \mathbf{R}_i \\ \\ \partial_t (\rho_e \boldsymbol{u}_e) + \nabla \cdot (\rho_e \boldsymbol{u}_e \otimes \boldsymbol{u}_e) + \nabla p_e = \sigma_e \mathbf{E} + \boldsymbol{J}_e \times \boldsymbol{B} - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_e + \mathbf{R}_e \\ \\ \partial_t \rho_i \epsilon_i + \nabla \cdot (\rho_i \epsilon_i \boldsymbol{u}_i + p_i \boldsymbol{u}_i) + \nabla \cdot \left(\mathbf{q}_i + \overline{\overline{\boldsymbol{\Pi}}}_i \cdot \boldsymbol{u}_i \right) = \sigma_i \boldsymbol{u}_i \cdot \boldsymbol{E} + \mathbf{R}_i \cdot \boldsymbol{u}_i + Q_{\Delta_i} \\ \\ \partial_t \rho_e \epsilon_e + \nabla \cdot (\rho_e \epsilon_e \boldsymbol{u}_e + p_e \boldsymbol{u}_e) + \nabla \cdot \left(\mathbf{q}_e + \overline{\overline{\boldsymbol{\Pi}}}_e \cdot \boldsymbol{u}_e \right) = \sigma_e \boldsymbol{u}_e \cdot \boldsymbol{E} + \mathbf{R}_e \cdot \boldsymbol{u}_e + Q_{\Delta_e} \end{cases} \tag{1.4}$$

and

$$\begin{cases} \frac{1}{c^2} \partial_t \boldsymbol{E} - \nabla \times \boldsymbol{B} = -\mu_0 \boldsymbol{J} \\ \\ \partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E} \\ \\ \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \cdot \boldsymbol{E} = \frac{1}{\varepsilon_0} \sigma \end{cases} \tag{1.5}$$

with $c = (\varepsilon_0 \mu_0)^{-\frac{1}{2}}$, $\sigma_{i,e} = q_{i,e} n_{i,e}$, the current for each species $\boldsymbol{J}_{i,e} = q_{i,e} n_{i,e} \boldsymbol{u}_{i,e}$ and density for each species $\rho_{i,e} = m_{i,e} n_{i,e}$. The charge for each particle is q, m correspond to the mass of the particle and n the density number. The friction coefficients between the species are defined by $\mathbf{R} = \mathbf{R}_i = -\mathbf{R}_e$.

$$Q_{\Delta_i} = 3 \frac{m_e n_e}{\tau_e m_i} (T_i - T_e), \quad Q_{\Delta_e} = -3 \frac{m_e n_e}{\tau_e m_i} (T_i - T_e) - \mathbf{R} \cdot (\boldsymbol{u}_i - \boldsymbol{u}_e)$$

Now we introduce the quantity for the total fluid.We define the density, momentum, pressure and current for the total fluid: $\rho = n_i m_i + n_e m_e$, $\rho \boldsymbol{u} = n_i m_i \boldsymbol{u}_i + n_e m_e \boldsymbol{u}_e$, $p = p_e + p_i$, $\Pi = \Pi_i + \Pi_e$, $\sigma = \sigma_i + \sigma_e$ and $\boldsymbol{J} = q_i n_i \boldsymbol{u}_i - q_e n_e \boldsymbol{u}_e$. Firstly we remark that $q_i = -q_e = e$.

Lemma

We assume that the boundary conditions are correctly choose. The equations (1.4)-(1.5) satisfy

$$\frac{d}{dt} \left(\int_\Omega (\rho_e \varepsilon_e + \rho_i \varepsilon_i) \, \mathrm{d}x + \frac{1}{2\mu_0 c^2} \int_\Omega | \boldsymbol{E} |^2 \, \mathrm{d}x + \frac{1}{2\mu_0} \int_\Omega | \boldsymbol{B} |^2 \, \mathrm{d}x \right) = 0$$

with $\rho_s \varepsilon_s = \int_{\mathbb{R}^3} m_s \frac{|\boldsymbol{v}_s|^2}{2} f_s \, \mathrm{d}v$.

Proof. Now we multiply the Ampere equation (equation which describe the evolution of the electric field in the Maxwell equation) by the electric field and we integrate to obtain

$$\frac{1}{2c^2} \partial_t \int_\Omega | \boldsymbol{E} |^2 \, \mathrm{d}x - \int_\Omega (\nabla \times \boldsymbol{B}) \cdot \boldsymbol{E} \, \mathrm{d}x = -\int_\Omega \mu_0 \boldsymbol{J} \cdot \boldsymbol{E} \, \mathrm{d}x$$

Now we consider the Faraday law (equation which describe the evolution of the Magnetic field). We multiply by the magnetic field and integrate.

$$\int_\Omega \partial_t \boldsymbol{B}\cdot\boldsymbol{B}\;\mathrm{d}x + \int_\Omega (\nabla\times\boldsymbol{E})\cdot\boldsymbol{B}\;\mathrm{d}x = 0$$
$$\frac{1}{2}\int_\Omega \partial_t \mid \boldsymbol{B}\mid^2\;\mathrm{d}x + \int_\Omega (\nabla\times\boldsymbol{B})\cdot\boldsymbol{E}\;\mathrm{d}x = 0$$

Coupling the two previous equations we obtain that

$$\frac{d}{dt}\left(\frac{1}{2\mu_0 c^2}\int_\Omega \mid \boldsymbol{E}\mid^2\;\mathrm{d}x + \frac{1}{2\mu_0}\int_\Omega \mid \boldsymbol{B}\mid^2\;\mathrm{d}x\right) = -\int_\Omega (\boldsymbol{J}\cdot\boldsymbol{E})\;\mathrm{d}x \tag{1.6}$$

In a second time we integrate and sum the two energy equations in the two fluid models

$$\sum_s\left[\int_\Omega \partial_t\rho_s\epsilon_s\;\mathrm{d}x + \int_\Omega \nabla\cdot(\rho_s\epsilon_s\boldsymbol{u}_s + p_s\boldsymbol{u}_s)\;\mathrm{d}x + \int_\Omega \nabla\cdot\left(\mathbf{q}_s + \overline{\overline{\boldsymbol{\Pi}}}_s\cdot\boldsymbol{u}_s\right)\;\mathrm{d}x\right]$$
$$= \int_\Omega (\sum_s \sigma_s\boldsymbol{u}_s)\cdot\boldsymbol{E}\;\mathrm{d}x + \sum_s\int_\Omega \mathbf{R}_s\cdot\boldsymbol{u}_s\;\mathrm{d}x + \sum_s\int_\Omega Q_{\Delta_s}\;\mathrm{d}x$$

Using the boundary conditions and the flux divergence theorem we obtain

$$\sum_s\left[\int_\Omega \partial_t\rho_s\epsilon_s\;\mathrm{d}x\right] = \int_\Omega (\sum_s \sigma_s\boldsymbol{u}_s)\cdot\boldsymbol{E}\;\mathrm{d}x + \sum_s\int_\Omega \mathbf{R}_s\cdot\boldsymbol{u}_s\;\mathrm{d}x + \sum_s\int_\Omega Q_{\Delta_s}\;\mathrm{d}x$$

By definition $\boldsymbol{J} = (\sum_s \sigma_s\boldsymbol{u}_s)$, $\sum_s Q_{\Delta_s} = -\mathbf{R}\cdot(\boldsymbol{u}_i - \boldsymbol{u}_e)$ consequently

$$\sum_s\left[\int_\Omega \partial_t\rho_s\epsilon_s\;\mathrm{d}x\right] = \int_\Omega \boldsymbol{J}\cdot\boldsymbol{E}\;\mathrm{d}x + \sum_s\int_\Omega \mathbf{R}_s\cdot\boldsymbol{u}_s\;\mathrm{d}x - \int_\Omega \mathbf{R}\cdot(\boldsymbol{u}_i - \boldsymbol{u}_e)\;\mathrm{d}x$$

Using that $\mathbf{R}_i = -\mathbf{R}_i = \mathbf{R}$ and coupling the previous equation and (1.6). ■

At the end of the previous section we have obtained the two fluid models. In this section we propose to write the nervy estimate for this model and after using some ordering derive the Extended MHD model and write the energy estimate for the model obtained (this is a single fluid model with two fluids effects). Before to construct the MHD we introduce the plasma parameters.

2 Definition of plasmas parameters

The first work is to defined the characteristic parameters and quantities for a Plasma in a Tokamak.

Frequencies:

- **Electron plasma frequency** : w_{pe}. Frequency of electron oscillation in the plasma..
- **Ion plasma frequency** : w_{pi}. Frequency of ion oscillation in the plasma.
- **Ion gyro frequency** : w_{ci}. The angular frequency of the circular motion of an ion in the plane perpendicular to the magnetic field.
- **Electron gyro frequency** : w_{ce}. The angular frequency of the circular motion of an ion in the plane perpendicular to the magnetic field.

- **Ion collision rate** : ν_i.
- **Electron collision rate**: ν_e.

Length:

- **Ion free path** : λ_i. Average distance between two subsequent collisions of the electron with plasma components
- **Electron free path** : λ_e. Average distance between two subsequent collisions of the ion with plasma components
- **Debye length** : λ_D. The scale over which electric fields are screened out by a redistribution of the electrons.
- **Ion gyro radius** : ρ_i^*. The radius of the circular motion of an ion in the plane perpendicular to the magnetic field.
- **Electron gyro radius** : ρ_e^*. The radius of the circular motion of an electron in the plane perpendicular to the magnetic field.

Velocities:

- **Ion thermal velocity** : v_{T_i}. Most probable velocity of an ion in a Maxwell-Boltzmann distribution.
- **Electron thermal velocity** : v_{T_e}. Most probable velocity of an electron in a Maxwell-Boltzmann distribution.
- **Sound speed** : c. The speed of the longitudinal waves resulting from the mass of the ions and the pressure of the electrons.
- **Alfvén wave** : v_A. The speed of the waves resulting from the mass of the ions and the restoring force of the magnetic field.

3 Derivation of Extended MHD using bi-fluid models

The two fluid model obtained admit the same energy estimate that the two species Vlasov-Maxwell equations. To verify that the following models which comes from to this one are correct. We propose to verify that we obtain the equivalent energy estimate for the other models. Actually this model describe the evolutions of the two fluids and is not close (for this we need to define the stress tensors the heat fluxes and the isotropic pressures). Now we propose to show that in the tokamak plasmas the assumptions of quasi neutrality is globally satisfy. This assumption allows to derive a one fluid model called "Extended MHD". In the following we will derive the model, gives the closures and the energy estimate.

$\tilde{t}$	$=$	tw_0	Π'	$=$	$\frac{\Pi}{\Pi_0}$
$\tilde{n}$	$=$	$\frac{n}{n_0}$	p'	$=$	$\frac{p}{p_0}$
$\tilde{\boldsymbol{u}}'$	$=$	$\frac{\boldsymbol{u}}{V_0}$	p_0	$=$	$n_0 m_i V_{T_i}^2$
∇'	$=$	$L\nabla$	q_0	$=$	$V_0 p_0$
$\boldsymbol{E}'$	$=$	$\frac{\boldsymbol{E}}{E_0}$	V_0	$=$	$\frac{V_A^2}{w_{pi}L}$
$\boldsymbol{B}'$	$=$	$\frac{\boldsymbol{B}}{B_0}$	J_0	$=$	$n_0 e V_0$
$\boldsymbol{J}'$	$=$	$\frac{\boldsymbol{J}}{J_0}$			

Table 1: Normalization

3.1 Derivation of the extended MHD

To derive the single fluid Extended MHD model we study the plasma in the quasi neutrality limit and the magnetostatic assumption. Firstly we propose to use dimensionless variables to obtain these two assumptions when the characteristic velocity of the plasma is smaller than the light velocity. Now we introduce the ordering and compute the dimensionless equations. We assume the normalization given in (1). We introduce $\varepsilon_1 = \frac{m_e}{m_i}$ and $\varepsilon_2 = \frac{V_0}{c}$ with $V_0 = \frac{V_A^2}{w_{ci}L}$ (V_A Alfven velocity, w_{ci} ion gyrofrequency).

Lemma

The model (1.4) with the previous rescaling is equivalent to the same fluid model with the new electromagnetic equations:

$$\begin{cases} \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + O(\varepsilon_2^2) \\ \partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E} \\ \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \cdot \boldsymbol{E} = \dfrac{1}{\varepsilon_0}\sigma \end{cases} \tag{3.7}$$

and $\boldsymbol{u} = \boldsymbol{u}_i + O(\varepsilon_1 + \varepsilon_2)$ and $\rho = \rho_i = m_i n_i + O(\varepsilon_1 + \varepsilon_2)$.

Proof. To simplify the notation we use t and not t' for exemple, but we use the dimensionless quantities. Using the recalling we obtain

$$\partial_t \boldsymbol{B} = -\frac{E_0}{w_0 L B_0} \nabla \times \boldsymbol{E}$$

and

$$\nabla \times \boldsymbol{B} = \frac{\mu_0 J_0 L}{B_0} \boldsymbol{J} + \frac{E_0 w_0 L}{B_0 c^2} \partial_t \boldsymbol{E}$$

Choosing $J_0 = \frac{B_0}{\mu L}$, $\boldsymbol{E}_0 = w_0 B_0 L$ and $w_0 = \frac{\boldsymbol{V}_0}{L}$ we obtain $\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E}$ and $\nabla \times \boldsymbol{B} = \boldsymbol{J} + \varepsilon_2^2 \partial_t \boldsymbol{E}$. Now we study the poisson equation which is

$$\nabla \cdot \boldsymbol{E} = \frac{\sigma L}{\varepsilon_0 E_0}$$

by definition we have $V_0 = \frac{J_0}{n_0 e} = \frac{V_A^2}{w_{ci} L}$ with $V_A^2 = \frac{B_0^2}{\mu_0 m_i n_0}$ and $w_{ci} = \frac{e B_0}{m_i}$. Now we consider that

$$\sigma = \frac{\varepsilon_0 E_0}{L} \nabla \cdot \boldsymbol{E}$$

using the relation $c = (\varepsilon_0 \mu_0)^{-\frac{1}{2}}$ and the definition of E_0 we obtain that $\frac{\varepsilon_0 E_0}{L} = \frac{w_0 B_0 L}{\mu_0 c^2 L} = \frac{V_0 B_0}{\mu_0 L c^2}$ by definition of w_0. To finish we use the relation with J_0 to obtain that $\sigma = (\nabla \times \boldsymbol{E}) n_0 e \frac{V_0^2}{c^2}$. By definition of σ and since $\nabla \times \boldsymbol{E} = O(1)$ we have

$$\frac{n_i - n_e}{n} \approx O\left(\frac{V_0^2}{c^2}\right) \approx \varepsilon_2^2$$

This is the quasi neutral approximation. Now we consider ρ which are equal to

$$\rho = m_i n_i + m_e n_e = m_i (n_i + \varepsilon_1 n_e)$$

Since $n = n_i = n_e + O(\varepsilon_2^2)$, $\rho = m_i n = m_i n_i + O(\varepsilon^1) + O(\varepsilon_2^2)$. Now we study the velocity

$$\boldsymbol{u} = \frac{m_i n_i \boldsymbol{u}_i + m_e n_e \boldsymbol{u}_e}{\rho} = \frac{m_i (n \boldsymbol{u}_i + \varepsilon_1 n \boldsymbol{u}_e) + O(\varepsilon_2^2)}{\rho} \approx \boldsymbol{u}_i + O(\varepsilon_1 + \varepsilon_2)$$

■

Lemma

When ε_1 and ε_2 tend to zero we obtain the following equation called Extended MHD:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0 \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \boldsymbol{J} \times \boldsymbol{B} - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}} \\ \partial_t p_i + \boldsymbol{u} \cdot \nabla p_i + \gamma p_i \nabla \cdot \boldsymbol{u} + \nabla \cdot \mathbf{q}_i + \overline{\overline{\boldsymbol{\Pi}}}_i : \nabla \boldsymbol{u} = 3 \frac{\rho_e}{\tau_e m_i} (T_i - T_e) \\ \partial_t p_e + \boldsymbol{u} \cdot \nabla p_e + \gamma p_e \nabla \cdot \boldsymbol{u} + \nabla \cdot \mathbf{q}_e + \overline{\overline{\boldsymbol{\Pi}}}_e : \nabla \boldsymbol{u} \\ = \frac{m_i}{\rho e} \boldsymbol{J} \cdot \left(\nabla p_e - \gamma p_e \frac{\nabla \rho}{\rho} \right) \overline{\overline{\boldsymbol{\Pi}}}_e : \nabla \left(\frac{m_i}{e \rho} \boldsymbol{J} \right) - 3 \frac{\rho_e}{\tau_e m_i} (T_i - T_e) + \eta |\boldsymbol{J}|^2 \\ \partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E} \\ \boldsymbol{E} = \left(-\boldsymbol{u} \times \boldsymbol{B} + \eta \boldsymbol{J} - \frac{m_i}{\rho e} \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_e - \frac{m_i}{\rho e} \nabla p_e + \frac{m_i}{\rho e} (\boldsymbol{J} \times \boldsymbol{B}) \right) \\ \mu_0 \nabla \times \boldsymbol{B} = \boldsymbol{J} \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases} \tag{3.8}$$

with $\boldsymbol{u} = \boldsymbol{u}_i$ and $\rho = \rho_i = m_i n_i$.

Proof. We begin to estimate the electric field with the other variable. This relation is called Generalized Ohm law.

Generalized Ohm Law

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \eta \boldsymbol{J} - \frac{m_i}{\rho e}\nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_e + \frac{m_i}{\rho e}\boldsymbol{J} \times \boldsymbol{B} - \frac{m_i}{\rho e}\nabla p_e.$$

We begin with the two fluids model. Taking the masse and velocity electronic equations we obtain

$$m_e\left(\partial_t(n_e\boldsymbol{u}_e) + \nabla \cdot (n_e\boldsymbol{u}_e \otimes \boldsymbol{u}_e)\right) + \nabla p_e = -en_e\boldsymbol{E} + \boldsymbol{J}_e \times \boldsymbol{B} - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_e + \boldsymbol{R}_e$$

we multiply the previous equation by $-e$ and we define $\boldsymbol{J}_e = -en_e\boldsymbol{u}_e$, we obtain

$$m_e\partial_t\boldsymbol{J}_e + m_e\nabla \cdot (\boldsymbol{J}_e \otimes \boldsymbol{u}_e) = e\nabla p_e + e^2 n_e\boldsymbol{E} - e\boldsymbol{J}_e \times \boldsymbol{B} + e\nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_e - \boldsymbol{R}_e$$

since $\boldsymbol{J}_e = -en_e\boldsymbol{u}_e$ we obtain at the end

$$\frac{m_e}{e^2 n_e}\left(\partial_t\boldsymbol{J}_e + \nabla \cdot (\boldsymbol{J}_e \otimes \boldsymbol{u}_e)\right) = \boldsymbol{E} + \boldsymbol{u}_e \times \boldsymbol{B} + \frac{1}{en_e}\nabla p_e + \frac{1}{en_e}\nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_e - \frac{1}{en_e}\boldsymbol{R}_e$$

Since the masse of the electron is close to zero, we neglected the terms multiply by m_e. Using that $\boldsymbol{u}_i \approx \boldsymbol{u}$, $n = n_e = \frac{\rho}{m_i}$ and $\boldsymbol{u}_e = \boldsymbol{u} - \frac{m_i}{\rho e}\boldsymbol{J}$, we obtain

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \frac{m_i}{\rho e}\boldsymbol{J} \times \boldsymbol{B} - \frac{m_i}{\rho e}\nabla p_e - \frac{m_i}{\rho e}\nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_e + \frac{m_i}{\rho e}\boldsymbol{R}_e.$$

To finish we use that $\boldsymbol{R} = -\boldsymbol{R}_e = -\eta\frac{e}{m_i}\rho\boldsymbol{J}$. Now we have obtain the Generalized Ohm law. We can derive the final model. Summing the equations on ρ_e and ρ_i we obtain

Mass equation

$$\partial_t\rho + \nabla \cdot (\rho\boldsymbol{u}) = 0 \tag{3.9}$$

Multiplying the momentum equations by the mass of the particles and summing these equations (we have $\boldsymbol{R}_e = -\boldsymbol{R}_i$.) we obtain the following approximation

$$\partial_t(\rho\boldsymbol{u}) + \nabla \cdot (\rho\boldsymbol{u}_i \otimes \boldsymbol{u}_i) + \nabla \cdot (\rho\boldsymbol{u}_e \otimes \boldsymbol{u}_e) + \nabla p = \rho(n_i - n_e)\boldsymbol{E} + \rho(n_i\boldsymbol{u}_i - n_e\boldsymbol{u}_e) \times \boldsymbol{B} - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}$$

Now we use that $\boldsymbol{u}_e = \boldsymbol{u} + \frac{\rho_i}{\rho_e}(\boldsymbol{u} - \boldsymbol{u}_i)$ and $\boldsymbol{u}_i = \boldsymbol{u} + \frac{\rho_e}{\rho_i}(\boldsymbol{u} - \boldsymbol{u}_e)$. These relation between the velocities gives

$$\partial_t(\rho\boldsymbol{u}) + \nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla\cdot(\rho_e\boldsymbol{u}_i\otimes(\boldsymbol{u}-\boldsymbol{u}_e) + \rho_i\boldsymbol{u}_e\otimes(\boldsymbol{u}-\boldsymbol{u}_i)) + \nabla p = \rho(n_i-n_e)\boldsymbol{E} + \rho(n_i\boldsymbol{u}_i - n_e\boldsymbol{u}_e)\times\boldsymbol{B} - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}$$

Now we use that $\boldsymbol{u} = \boldsymbol{u}_i + O(\varepsilon_1 + \varepsilon_2)$ we obtain

$$\partial_t(\rho\boldsymbol{u}) + \nabla \cdot (\rho\boldsymbol{u} \otimes \boldsymbol{u}) + \frac{m_i\rho_e}{e\rho}\boldsymbol{u} \otimes \boldsymbol{J} + \nabla p = \rho(n_i - n_e)\boldsymbol{E} + \rho(n_i\boldsymbol{u}_i - n_e\boldsymbol{u}_e) \times \boldsymbol{B} - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}} + O(\varepsilon_1 + \varepsilon_2)$$

Using the definition of $\boldsymbol{J}$, the fact that $\frac{m_i\rho_e}{e\rho}\boldsymbol{u}\otimes\boldsymbol{J}$ is homogeneous to $O(\varepsilon_1+\varepsilon_2)$, since $n_i - n_e = O(\varepsilon_2^2)$ gives the following momentum equation

$$\partial_t\rho\boldsymbol{u} + \nabla \cdot (\rho\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \boldsymbol{J} \times \boldsymbol{B} - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}} + O(\varepsilon_1 + \varepsilon_2)$$

Using the equation on the mass and neglecting the term homogeneous to ε we obtain the following equation on the velocity

Momentum equation

$$\rho\partial_t \boldsymbol{u} + \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{J}\times\boldsymbol{B} - \nabla\cdot\overline{\overline{\boldsymbol{\Pi}}} \tag{3.10}$$

Now we consider the pressure equation. We begin by write the equation for each velocity and multiply this equation by this velocity to obtain

$$\partial_t(\rho_i|\boldsymbol{u}_i|^2) + \nabla\cdot(\rho|\boldsymbol{u}_i|^2\boldsymbol{u}_i) + \boldsymbol{u}_i\nabla p = \sigma_i\boldsymbol{u}_i\cdot\boldsymbol{E} + (\boldsymbol{J}_i\times\boldsymbol{B})\cdot\boldsymbol{u}_i - \nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}_i + \boldsymbol{R}_i\cdot\boldsymbol{u}_i$$

idem for the electronic case. Since $\boldsymbol{J}_{i,e} = \sigma_{i,e}\boldsymbol{u}_{i,e}$ then $(\boldsymbol{J}_{i,e}\times\boldsymbol{B})\cdot\boldsymbol{u}_{i,e} = 0$. Now we subtract the previous equation to the equation of ion energy (the same for electronic case). We obtain

$$\frac{1}{\gamma-1}\partial_t p_i + \frac{1}{\gamma-1}\nabla\cdot(p_i\boldsymbol{u}_i) + \nabla\cdot(p_i\boldsymbol{u}_i) - \boldsymbol{u}_i\cdot\nabla p_i + \nabla\cdot\mathbf{q}_i + \overline{\overline{\boldsymbol{\Pi}}}_i : \nabla\boldsymbol{u}_i = Q_{\Delta_i}$$
$$\frac{1}{\gamma-1}\partial_t p_e + \frac{1}{\gamma-1}\nabla\cdot(p_e\boldsymbol{u}_e) + \nabla\cdot(p_e\boldsymbol{u}_e) - \boldsymbol{u}_i\cdot\nabla p_e + \nabla\cdot\mathbf{q}_e + \overline{\overline{\boldsymbol{\Pi}}}_e : \nabla\boldsymbol{u}_e = Q_{\Delta_e}$$

We obtain the same equation for the electronic pressure. We sum the two equations to obtain after small approximations for the source term :

$$\frac{1}{\gamma-1}\partial_t p_i + \frac{\gamma}{\gamma-1}\nabla\cdot(p_i\boldsymbol{u}_i) - \boldsymbol{u}_i\cdot\nabla p_i + \nabla\cdot\mathbf{q}_i + \overline{\overline{\boldsymbol{\Pi}}}_i : \nabla\boldsymbol{u}_i = Q_{\Delta_i}$$
$$\frac{1}{\gamma-1}\partial_t p_e + \frac{\gamma}{\gamma-1}\nabla\cdot(p_e\boldsymbol{u}_e) - \boldsymbol{u}_e\cdot\nabla p_e + \nabla\cdot\mathbf{q}_e + \overline{\overline{\boldsymbol{\Pi}}}_e : \nabla\boldsymbol{u}_e = Q_{\Delta_e}$$

Now we replace $\boldsymbol{u}_e$ by $\boldsymbol{u}_i - \frac{m_i}{e\rho}\boldsymbol{J} = \boldsymbol{u} - \frac{m_i}{e\rho}\boldsymbol{J}$. Using that $\boldsymbol{u} = \boldsymbol{u}_i + O(\varepsilon_1+\varepsilon_2)$ we obtain

$$\frac{1}{\gamma-1}\partial_t p_i + \frac{1}{\gamma-1}\boldsymbol{u}\cdot\nabla p_i + \frac{\gamma}{\gamma-1}p_i\nabla\cdot\boldsymbol{u} + \nabla\cdot\boldsymbol{q}_i + \overline{\overline{\boldsymbol{\Pi}}}_i : \nabla\boldsymbol{u} = 3\frac{\rho_e}{\tau_e m_i}(T_i - T_e)$$
$$\frac{1}{\gamma-1}\partial_t p_e + \frac{1}{\gamma-1}\boldsymbol{u}\cdot\nabla p_e + \frac{\gamma}{\gamma-1}p_e\nabla\cdot\boldsymbol{u} - \frac{\gamma}{\gamma-1}\frac{m_i}{e}\nabla\cdot(\frac{p_e}{\rho}\boldsymbol{J}) - \frac{1}{\gamma-1}\frac{m_i}{\rho e}\boldsymbol{J}\cdot\nabla p_e = \overline{\overline{\boldsymbol{\Pi}}}_e : \nabla\left(\frac{m_i}{e\rho}\boldsymbol{J}\right)$$
$$+ 3\frac{\rho_e}{\tau_e m_i}(T_i - T_e) - \boldsymbol{R}\cdot(\boldsymbol{u}_i - \boldsymbol{u}_e)$$

Now we study the term

$$-\frac{\gamma}{\gamma-1}\frac{m_i}{e}\nabla\cdot\left(\frac{p_e}{\rho}\boldsymbol{J}\right) - \frac{1}{\gamma-1}\frac{m_i}{\rho e}\boldsymbol{J}\cdot\nabla p_e$$

which is equal to

$$-\frac{m_i}{e\rho}\frac{1}{\gamma-1}\boldsymbol{J}\cdot\nabla p_e - \frac{\gamma}{\gamma-1}\frac{m_i}{e}p_e\boldsymbol{J}\cdot\nabla\left(\frac{1}{\rho}\right)$$

At this end we obtain

$$\frac{1}{\gamma-1}\partial_t p_i + \frac{1}{\gamma-1}\boldsymbol{u}\cdot\nabla p_i + \frac{\gamma}{\gamma-1}p_i\nabla\cdot\boldsymbol{u} + \nabla\cdot\boldsymbol{q}_i + \overline{\overline{\boldsymbol{\Pi}}}_i : \nabla\boldsymbol{u} = 3\frac{\rho_e}{\tau_e m_i}(T_i - T_e)$$
$$\frac{1}{\gamma-1}\partial_t p_e + \frac{1}{\gamma-1}\boldsymbol{u}\cdot\nabla p_e + \frac{\gamma}{\gamma-1}p_e\nabla\cdot\boldsymbol{u} - \frac{1}{\gamma-1}\frac{m_i}{\rho e}\boldsymbol{J}\cdot\left(\nabla p_e - \gamma p_e\frac{\nabla\rho}{\rho}\right) = \overline{\overline{\boldsymbol{\Pi}}}_e : \nabla\left(\frac{m_i}{e\rho}\boldsymbol{J}\right)$$
$$+ 3\frac{\rho_e}{\tau_e m_i}(T_i - T_e) - \mathbf{R}\cdot(\boldsymbol{u}_i - \boldsymbol{u}_e)$$

To finish we remark that

$$-\boldsymbol{R}\cdot(\boldsymbol{u}_i-\boldsymbol{u}_e)=-\frac{m_i}{e\rho}\boldsymbol{J}\cdot\boldsymbol{R}=\eta|\boldsymbol{J}|^2$$

Consequently

Pressure equation

$$\frac{1}{\gamma-1}\partial_t p_i+\frac{1}{\gamma-1}\boldsymbol{u}\cdot\nabla p_i+\frac{\gamma}{\gamma-1}p_i\nabla\cdot\boldsymbol{u}+\nabla\cdot\boldsymbol{q}_i+\overline{\overline{\boldsymbol{\Pi}}}_i:\nabla\boldsymbol{u}=3\frac{\rho_e}{\tau_e m_i}(T_i-T_e)$$

$$\frac{1}{\gamma-1}\partial_t p_e+\frac{1}{\gamma-1}\boldsymbol{u}\cdot\nabla p_e+\frac{\gamma}{\gamma-1}p_e\nabla\cdot\boldsymbol{u}+\nabla\cdot\boldsymbol{q}_e+\overline{\overline{\boldsymbol{\Pi}}}_e:\nabla\boldsymbol{u}$$
$$=\frac{1}{\gamma-1}\frac{m_i}{\rho e}\boldsymbol{J}\cdot\left(\nabla p_e-\gamma p_e\frac{\nabla\rho}{\rho}\right)=\overline{\overline{\boldsymbol{\Pi}}}_e:\nabla\left(\frac{m_i}{e\rho}\boldsymbol{J}\right)-3\frac{\rho_e}{\tau_e m_i}(T_i-T_e)+\eta|\boldsymbol{J}|^2$$

■

The model obtained is called "Extended MHD". This system contained the Ideal MHD (black terms), the resistive MHD (blue and black terms) and some two fluid effects (red terms). Later in this documents we will prove how obtain these models with the Extended MHD. This model is study in the code M3D-C1 [JARDIN]-[FERRARO 2008].

To finish the modelization of the plasma by this extended Model we will define the closures which gives the form of the stress tensors and the heat fluxes.

3.2 Braginskii Closure

In this subsection we will define the classical and more complexe closures used for the Extended MHD code. The expression of the stress tensors are simplified in the other models defined later.

3.2.1 Heat flux Braginskii closure

We begin by details the heat flux $\mathbf{q}$. The Branginskii closure gives the following writing of the ionic heating flux

Heat flux

$$\mathbf{q}_i=-n_i\chi_\parallel^i\nabla_\parallel T_i-n_i\chi_c^i\nabla_c T_i-n_i\chi_\perp^i\nabla_\perp T_i$$

and

$$\mathbf{q}_e=-n_e\chi_\parallel^e\nabla_\parallel T_e-n_e\chi_c^e\nabla_c T_e-n_e\chi_\perp^e\nabla_\perp T_e+....$$

with $\nabla_\parallel=\frac{\boldsymbol{B}}{\|\boldsymbol{B}\|}\cdot(\frac{\boldsymbol{B}}{\|\boldsymbol{B}\|}\cdot\nabla)$, $\nabla_\perp=\frac{\boldsymbol{B}}{|\boldsymbol{B}|}\times(\frac{\boldsymbol{B}}{|\boldsymbol{B}|}\times\nabla)$ and $\nabla_c=\frac{\boldsymbol{B}}{|\boldsymbol{B}|}\times\nabla$. The Braginskii closure gives the following coefficient

$$\chi_\parallel^{i,e}\approx\varepsilon_\parallel^2,\quad \chi_c^{i,e}\approx\varepsilon_\perp,\quad \chi_\perp^{i,e}\approx\varepsilon_\perp^2$$

with $\varepsilon_\parallel=\frac{\lambda_{i,e}}{L}$ and $\varepsilon_\perp=\frac{\rho_{i,e}^*}{L_\perp}$ (L and $L_\perp$) the parallel and perpendicular characteristic length.

Scale regime is the Tokamak

$$\varepsilon_\perp = \frac{\rho^*_{i,e}}{L_\perp} << \varepsilon_\| = \frac{\lambda_{i,e}}{L}$$

These scaling shows that the diffusion operator is strongly anistropic.

3.2.2 Stress tensor Braginskii closure

In the different paper the closure used is proposed by Braginskii [BRAGINSKII 1965]. This closure assume that a small armor radius, a small parallel mean fee path and large collision frequency if you compare to the characteristic frequency. In This closure the tensor is decomposed between three parts : the parallel, cross and perpendicular part defined by

- $\overline{\overline{\overline{\mathbf{\Pi}}}}_\| = -\frac{3}{2}\eta_0 p_i(\boldsymbol{B}\cdot\boldsymbol{W}\cdot\boldsymbol{B})(I_d - \boldsymbol{B}\boldsymbol{B}) = \frac{1}{\nu}\overline{\overline{\overline{\mathbf{\Pi}}}}^s_\|$
- $\overline{\overline{\overline{\mathbf{\Pi}}}}_c = -\frac{1}{2}\eta_3 p_i(\boldsymbol{B}\times\boldsymbol{W}\cdot(I_d + 3\boldsymbol{B}\boldsymbol{B}) + transpose) = \frac{1}{w_{ci}}\overline{\overline{\overline{\mathbf{\Pi}}}}^s_c$
- $\overline{\overline{\overline{\mathbf{\Pi}}}}_\perp = -\eta_1 p_i((I_d - \boldsymbol{B}\boldsymbol{B})\cdot\boldsymbol{W}\cdot(I_d - \boldsymbol{B}\boldsymbol{B}) - \frac{1}{2}(I_d - \boldsymbol{B}\boldsymbol{B})(I_d - \boldsymbol{B}\boldsymbol{B}) : \boldsymbol{W} + 4((I_d - \boldsymbol{B}\boldsymbol{B})\cdot\boldsymbol{W}\cdot \boldsymbol{B}\boldsymbol{B} + transpose)) = \frac{\nu}{w_{ci}^2}\overline{\overline{\overline{\mathbf{\Pi}}}}^s_\perp$

with $\boldsymbol{W} = \nabla\boldsymbol{u} + \nabla\boldsymbol{u}^t - \frac{2}{3}I_d\nabla\cdot\boldsymbol{u}$, $\eta_0 = 0.96\frac{p_i}{\nu}$, $\eta_1 = \frac{3}{10}\frac{p_i\nu}{w_{ci}}$ and $\eta_3 = \frac{p_i}{2w_{ci}}$

3.3 Energy estimate for the extended MHD

Lemma

The total energy for the MHD is given by with $p = \rho T$ and $\gamma = \frac{5}{3}$. The conservation law for the total energy is given by

$$E = \rho\frac{|\boldsymbol{u}|^2}{2} + \frac{|\boldsymbol{B}|^2}{2} + \frac{1}{\gamma - 1}(p_e + p_i). \tag{3.11}$$

with $p = \rho T$ and $\gamma = \frac{5}{3}$. The conservation law for the total energy is given by

$$\partial_t E + \nabla\cdot\left[\boldsymbol{u}\left(\rho\frac{|\boldsymbol{u}|^2}{2} + \frac{\gamma}{\gamma-1}(p_e + p_i)\right) - (\boldsymbol{u}\times\boldsymbol{B})\times\boldsymbol{B}\right] \tag{3.12}$$

$$+\nabla\cdot\left[\frac{m_i}{\rho e}\left((\boldsymbol{J}\times\boldsymbol{B})\times\boldsymbol{B} - \nabla p_e\times\boldsymbol{B} - (\nabla\cdot\overline{\overline{\overline{\mathbf{\Pi}}}}_e)\times\boldsymbol{B} - \frac{\gamma}{\gamma-1}p_e\boldsymbol{J} - \boldsymbol{J}\cdot\overline{\overline{\overline{\mathbf{\Pi}}}}_e\right)\right] \tag{3.13}$$

$$+\nabla\cdot\mathbf{q} + \nabla\cdot(\overline{\overline{\overline{\mathbf{\Pi}}}}\cdot\boldsymbol{u}) + \eta\nabla\cdot(\boldsymbol{J}\times\boldsymbol{B}) = 0 \tag{3.14}$$

If we integrate in space the previous energy conservation laws we obtain that the energy is conserved in time.

Proof. Firstly we define the total pressure $p = p_e + p_i$. Secondly adding the two pressure equation we obtain the following equation on the total pressure

$$\frac{1}{\gamma-1}\partial_t p+\frac{1}{\gamma-1}\boldsymbol{u}\cdot\nabla p+\frac{\gamma}{\gamma-1}p\nabla\cdot\boldsymbol{u}+\nabla\cdot\mathbf{q}=\frac{1}{\gamma-1}\frac{m_i}{e\rho}\boldsymbol{J}\cdot\left(\nabla p_e-\gamma p_e\frac{\nabla\rho}{\rho}\right)-\overline{\overline{\boldsymbol{\Pi}}}:\nabla\boldsymbol{u}+\eta|\boldsymbol{J}|^2+\Pi_e:\nabla\left(\frac{m_i}{e\rho}\boldsymbol{J}\right)$$

We multiply the mass equation by $\frac{|\boldsymbol{u}|^2}{2}$, the velocity equation by $\boldsymbol{u}$, the pressure equation by $\frac{1}{\gamma-1}$ and the Magnetic equation by $\boldsymbol{B}$. We obtain

$$\begin{cases}
\frac{|\boldsymbol{u}|^2}{2}\left(\partial_t\rho+\nabla\cdot(\rho\boldsymbol{u})\right)= \\
\boldsymbol{u}\cdot\left(\rho\partial_t\boldsymbol{u}+\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla(p)=\boldsymbol{J}\times\boldsymbol{B}-\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}\right) \\
\partial_t\frac{p}{\gamma-1}+\frac{1}{\gamma-1}\boldsymbol{u}\cdot\nabla p+\frac{\gamma}{\gamma-1}p\nabla\cdot\boldsymbol{u}+\nabla\cdot\mathbf{q}=\frac{m_i}{(\gamma-1)e\rho}\boldsymbol{J}\cdot\left(\nabla p_e-\gamma p_e\frac{\nabla\rho}{\rho}\right) \\
-\overline{\overline{\boldsymbol{\Pi}}}:\nabla\boldsymbol{u}+\eta|\boldsymbol{J}|^2+\Pi_e:\nabla\left(\frac{m_i}{e\rho}\boldsymbol{J}\right) \\
\boldsymbol{B}\cdot\partial_t\boldsymbol{B}=\boldsymbol{B}\cdot\left(\nabla\times\left(\boldsymbol{u}\times\boldsymbol{B}-\eta\boldsymbol{J}+\frac{m_i}{\rho e}\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}_e+\frac{m_i}{\rho e}\nabla P_e-\frac{m_i}{\rho e}(\boldsymbol{J}\times\boldsymbol{B})\right)\right) \\
\nabla\times\boldsymbol{B}=\boldsymbol{J} \\
\nabla\cdot\boldsymbol{B}=0
\end{cases} \tag{3.15}$$

Firstly using the velocity equation multiply by $\boldsymbol{u}$ and the mass equation multiply by $\frac{|\boldsymbol{u}|^2}{2}$ to obtain the equation on the kinetic energy

$$\partial_t(\rho\boldsymbol{u})+\nabla\cdot\left(\rho|\boldsymbol{u}|^2\boldsymbol{u}\right)+\boldsymbol{u}\cdot\nabla p=(\boldsymbol{J}\times\boldsymbol{B})\cdot\boldsymbol{u}-(\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}})\cdot\boldsymbol{u}$$

Adding this equation to the pressure equation and the magnetic field equation multiply by $\boldsymbol{B}$. We obtain

$$\begin{aligned}
\partial_t E=&-\frac{|\boldsymbol{u}|^2}{2}\nabla\cdot(\rho\boldsymbol{u})-\rho|\boldsymbol{u}|^2\cdot\nabla\boldsymbol{u}-\frac{\gamma}{\gamma-1}\boldsymbol{u}\cdot\nabla p-\frac{\gamma}{\gamma-1}p\nabla\cdot\boldsymbol{u}+\boldsymbol{B}(\nabla\times\cdot(\boldsymbol{u}\times\boldsymbol{B}))+\boldsymbol{u}\cdot(\boldsymbol{J}\times\boldsymbol{B}) \\
&+\nabla\times\left(\frac{m_i}{\rho e}\nabla p_e\right)\cdot\boldsymbol{B}-\nabla\times\left(\frac{m_i}{\rho e}(\boldsymbol{J}\times\boldsymbol{B})\right)\cdot\boldsymbol{B}+\frac{m_i}{(\gamma-1)e\rho}\boldsymbol{J}\cdot\left(\nabla p_e-\gamma p_e\frac{\nabla\rho}{\rho}\right) \\
&+\overline{\overline{\boldsymbol{\Pi}}}_e:\nabla\left(\frac{m_i}{e\rho}\boldsymbol{J}\right)+\nabla\times\left(\frac{m_i}{e\rho}\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}_e\right)\cdot\boldsymbol{B} \\
&+\nabla\cdot\mathbf{q}-\eta(\nabla\times\boldsymbol{J}\cdot\boldsymbol{B})+\eta|\boldsymbol{J}|^2-(\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}})\cdot\boldsymbol{u}-\overline{\overline{\boldsymbol{\Pi}}}:\nabla\boldsymbol{u}
\end{aligned}$$

Using the same argument that for the resistive case

$$\begin{aligned}
\partial_t E=&-\nabla\cdot\left((\rho\frac{|\boldsymbol{u}|^2}{2}+\frac{\gamma}{\gamma-1}p)\boldsymbol{u}\right)+\nabla\cdot((\boldsymbol{u}\times\boldsymbol{B})\times\boldsymbol{B}) \\
&+\nabla\times\left(\frac{m_i}{\rho e}\nabla P_e\right)\cdot\boldsymbol{B}-\nabla\times\left(\frac{m_i}{\rho e}(\boldsymbol{J}\times\boldsymbol{B})\right)\cdot\boldsymbol{B}+\frac{m_i}{(\gamma-1)e\rho}\boldsymbol{J}\cdot\left(\nabla p_e-\gamma p_e\frac{\nabla\rho}{\rho}\right) \\
&+\overline{\overline{\boldsymbol{\Pi}}}_e:\nabla\left(\frac{m_i}{e\rho}\boldsymbol{J}\right)+\nabla\times\left(\frac{m_i}{e\rho}\left(\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}_e\right)\right)\cdot\boldsymbol{B} \\
&+\nabla\cdot\mathbf{q}-\eta(\nabla\times\boldsymbol{J}\cdot\boldsymbol{B})+\eta|\boldsymbol{J}|^2-(\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}})\cdot\boldsymbol{u}-\overline{\overline{\boldsymbol{\Pi}}}:\nabla\boldsymbol{u}
\end{aligned}$$

Now we consider the terms linked to the electronic stress tensor. We have

$$\begin{aligned}
\nabla\times\left(\frac{m_i}{e\rho}\left(\nabla\cdot\overline{\overline{\mathbf{\Pi}}}_e\right)\right)\cdot\boldsymbol{B} &= \nabla\cdot\left(\frac{m_i}{e\rho}\left(\nabla\cdot\overline{\overline{\mathbf{\Pi}}}_e\right)\times\boldsymbol{B}\right)+\frac{m_i}{e\rho}\left(\nabla\cdot\overline{\overline{\mathbf{\Pi}}}_e\right)\cdot\boldsymbol{J}\\
&= \nabla\cdot\left(\frac{m_i}{e\rho}\left(\nabla\cdot\overline{\overline{\mathbf{\Pi}}}_e\right)\times\boldsymbol{B}\right)+\nabla\cdot\left(\frac{m_i}{\rho e}\Pi_e\cdot\boldsymbol{J}\right)-\overline{\overline{\mathbf{\Pi}}}_e:\nabla\left(\frac{m_i}{e\rho}\boldsymbol{J}\right)
\end{aligned}$$

To obtain this we have use $\nabla\cdot(\mathbf{a}\times\mathbf{b})=(\nabla\times\mathbf{a})\cdot\mathbf{b}-\mathbf{a}(\nabla\times\mathbf{b})$. Consequently we have

$$\begin{aligned}
\partial_t E = &-\nabla\cdot\left((\rho\frac{|\boldsymbol{u}|^2}{2}+\frac{\gamma}{\gamma-1}p)\boldsymbol{u}\right)+\nabla\cdot((\boldsymbol{u}\times\boldsymbol{B})\times\boldsymbol{B})\\
&\nabla\cdot\left(\frac{m_i}{e\rho}\left(\nabla\cdot\overline{\overline{\mathbf{\Pi}}}_e\right)\times\boldsymbol{B}\right)+\nabla\cdot\left(\frac{m_i}{\rho e}\overline{\overline{\mathbf{\Pi}}}_e\cdot\boldsymbol{J}\right)\\
&+\nabla\times\left(\frac{m_i}{\rho e}\nabla P_e\right)\cdot\boldsymbol{B}-\nabla\times\left(\frac{m_i}{\rho e}\left(\boldsymbol{J}\times\boldsymbol{B}\right)\right)\cdot\boldsymbol{B}+\frac{m_i}{(\gamma-1)e\rho}\boldsymbol{J}\cdot\left(\nabla p_e-\gamma p_e\frac{\nabla\rho}{\rho}\right)\\
&+\nabla\cdot\mathbf{q}-\eta\left(\nabla\times\boldsymbol{J}\cdot\boldsymbol{B}\right)+\eta|\boldsymbol{J}|^2-(\nabla\cdot\overline{\overline{\mathbf{\Pi}}})\cdot\boldsymbol{u}-\overline{\overline{\mathbf{\Pi}}}:\nabla\boldsymbol{u}
\end{aligned}$$

Now we use $\nabla\cdot(a\mathbf{b})=a\nabla\cdot\mathbf{b}+\nabla a\cdot\mathbf{b}$. Using the same relation we obtain that

$$\boldsymbol{B}\cdot\left(\nabla\times\left(\frac{m_i}{\rho e}\boldsymbol{J}\times\boldsymbol{B}\right)\right)=\nabla\cdot\left(\frac{m_i}{\rho e}\left(\boldsymbol{J}\times\boldsymbol{B}\right)\times\boldsymbol{B}\right)+\left(\frac{m_i}{\rho e}\boldsymbol{J}\times\boldsymbol{B}\right)\cdot(\nabla\times\boldsymbol{B})$$

The term $\left(\frac{m_i}{\rho e}\boldsymbol{J}\times\boldsymbol{B}\right)\cdot(\nabla\times\boldsymbol{B})$ is equal to zero since $\boldsymbol{J}=\nabla\times\boldsymbol{B}$ and $(\boldsymbol{J}\times\boldsymbol{B})\cdot\boldsymbol{J}=0$. We use also that

$$\nabla\times\left(\frac{m_i}{\rho e}\nabla P_e\right)=\nabla\cdot\left(\frac{m_i}{\rho e}\nabla p_e\times\boldsymbol{B}\right)+\frac{m_i}{\rho e}\nabla p_e\cdot\boldsymbol{J}$$

to obtain

$$\begin{aligned}
\partial_t E = &-\nabla\cdot\left((\rho\frac{|\boldsymbol{u}|^2}{2}+\frac{\gamma}{\gamma-1}p)\boldsymbol{u}\right)+\nabla\cdot((\boldsymbol{u}\times\boldsymbol{B})\times\boldsymbol{B})\\
&+\nabla\cdot\left(\frac{m_i}{e\rho}(\nabla\cdot\overline{\overline{\mathbf{\Pi}}}_e)\times\boldsymbol{B}\right)+\nabla\cdot\left(\frac{m_i}{e\rho}\overline{\overline{\mathbf{\Pi}}}_e\cdot\boldsymbol{J}\right)\\
&-\nabla\cdot\left(\frac{m_i}{e\rho}(\boldsymbol{J}\times\boldsymbol{B}-\nabla p_e)\times\boldsymbol{B}\right)\\
&+\frac{m_i}{(\gamma-1)e\rho}\boldsymbol{J}\cdot\left(\nabla p_e-\gamma p_e\frac{\nabla\rho}{\rho}\right)+\frac{m_i}{e\rho}\nabla p_e\cdot\boldsymbol{J}\\
&+\nabla\cdot\mathbf{q}-\eta(\nabla\times\boldsymbol{J}\cdot\boldsymbol{B})+\eta|\boldsymbol{J}|^2-(\nabla\cdot\overline{\overline{\mathbf{\Pi}}})\cdot\boldsymbol{u}-\overline{\overline{\mathbf{\Pi}}}:\nabla\boldsymbol{u}
\end{aligned}$$

Now we compute the term $T=\frac{m_i}{(\gamma-1)e\rho}\boldsymbol{J}\cdot\left(\nabla p_e-\gamma p_e\frac{\nabla\rho}{\rho}\right)+\frac{m_i}{e\rho}\nabla p_e\cdot\boldsymbol{J}$. We obtain

$$\begin{aligned}
T &= +\frac{m_i}{(\gamma-1)e\rho}\boldsymbol{J}\cdot\left(\nabla p_e-\gamma p_e\frac{\nabla\rho}{\rho}\right)+\frac{m_i}{e\rho}\nabla p_e\cdot\boldsymbol{J}\\
&= +\frac{m_i}{(\gamma-1)e\rho}\boldsymbol{J}\cdot\left(\nabla p_e-\gamma p_e\frac{\nabla\rho}{\rho}\right)+\frac{\gamma-1}{\gamma-1}\frac{m_i}{e\rho}\nabla p_e\cdot\boldsymbol{J}\\
&= +\frac{\gamma}{\gamma-1}\boldsymbol{J}\cdot\left(\frac{\nabla p_e}{\rho}\right)-\frac{\gamma}{\gamma-1}\boldsymbol{J}\cdot\left(p_e\frac{\nabla\rho}{\rho^2}\right)\\
&= +\frac{\gamma}{\gamma-1}\boldsymbol{J}\cdot\left(\frac{\nabla p_e}{\rho}\right)+\frac{\gamma}{\gamma-1}\boldsymbol{J}\cdot\left(p_e\nabla(\frac{1}{\rho})\right)\\
&= +\frac{\gamma}{\gamma-1}\boldsymbol{J}\cdot\nabla\left(\frac{p_e}{\rho}\right)=+\frac{\gamma}{\gamma-1}\nabla\cdot\left(\frac{p_e}{\rho}\boldsymbol{J}\right)
\end{aligned}$$

the last result is given by $\nabla \cdot \boldsymbol{J} = 0$.

$$\begin{aligned}\partial_t E = &- \nabla \cdot \left((\rho \frac{|\boldsymbol{u}|^2}{2} + \frac{\gamma}{\gamma - 1} p) \boldsymbol{u} \right) + \nabla \cdot ((\boldsymbol{u} \times \boldsymbol{B}) \times \boldsymbol{B}) \\ & \nabla \cdot \left(\frac{m_i}{e\rho} (\nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_e) \times \boldsymbol{B} \right) + \nabla \cdot \left(\frac{m_i}{e\rho} \overline{\overline{\boldsymbol{\Pi}}}_e \cdot \boldsymbol{J} \right) \\ & - \nabla \cdot \left(\frac{m_i}{e\rho} (\boldsymbol{J} \times \boldsymbol{B} - \nabla p_e) \times \boldsymbol{B} \right) + \frac{\gamma}{\gamma - 1} \nabla \cdot \left(\frac{p_e}{\rho} \boldsymbol{J} \right) \\ & + \nabla \cdot \mathbf{q} - \eta (\nabla \times \boldsymbol{J} \cdot \boldsymbol{B}) + \eta |\boldsymbol{J}|^2 - (\nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}) \cdot \boldsymbol{u} - \overline{\overline{\boldsymbol{\Pi}}} : \nabla \boldsymbol{u}\end{aligned}$$

To finish we use that

$$\nabla \times \boldsymbol{J} \cdot \boldsymbol{B} = \nabla \cdot (\boldsymbol{J} \times \boldsymbol{B}) + |\boldsymbol{J}|^2$$

The proof is finished. ■

The result show that we have an an energy (total physical energy) which is preserved. This result is interesting for two reason. Firstly this is a first step to obtain the stability of the model. Secondly the conservation of the energy is important at the physical level consequently it is a good criterion to validate the following reduced model that we have a energy balanced law close to this one.

4 Ordering for Extended MHD In Tokamak

In this section, we propose to define dimensionless equations for Extended MHD. Using this dimensionless equations, we will given ordering which is use in the Tokamak. In a following section we cill use this ordering to simplify a little the stress tensor and some other terms. We begin with the model with generalized Ohm law with full stress tensor. The ordering proposed in this part comes from to [SCHNACK 2006]-[SCHNACK]. Just for the ordering we consider the equation on the total pressure (but we using the electronic pressure equation we can find the ion and electron pressure equations).

Extended MHD with single pressure

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0 \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \boldsymbol{J} \times \boldsymbol{B} - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}} \\ \frac{1}{\gamma - 1} \partial_t p + \frac{1}{\gamma - 1} \boldsymbol{u} \cdot \nabla p + \frac{\gamma}{\gamma - 1} p \nabla \cdot \boldsymbol{u} + \nabla \cdot \mathbf{q} = \frac{1}{\gamma - 1} \frac{m_i}{e\rho} \boldsymbol{J} \cdot \left(\nabla p_e - \gamma p_e \frac{\nabla \rho}{\rho} \right) \\ - \overline{\overline{\boldsymbol{\Pi}}} : \nabla \boldsymbol{u} + \overline{\overline{\boldsymbol{\Pi}}}_e : \nabla \left(\frac{m_i}{e\rho} \boldsymbol{J} \right) + \eta |\boldsymbol{J}|^2 \\ \partial_t \boldsymbol{B} = - \nabla \times \boldsymbol{E} \\ \boldsymbol{E} = \left(- \boldsymbol{u} \times \boldsymbol{B} + \eta \boldsymbol{J} - \frac{m_i}{\rho e} \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_e - \frac{m_i}{\rho e} \nabla p_e + \frac{m_i}{\rho e} (\boldsymbol{J} \times \boldsymbol{B}) \right) \\ \mu_0 \nabla \times \boldsymbol{B} = \boldsymbol{J} \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases} \tag{4.16}$$

Using the same assumptions as before we obtain the ordering for the Extended MHD (4.16). Before to begin the ordering we define the important small parameters.

Small Parameters

- Time small parameter: $\varepsilon = \frac{w_0}{w_{ci}}$
- Velocity small parameter: $\xi = \frac{V_0}{V_{T_i}}$
- Spatial small parameter: $\delta = \frac{\rho_i^*}{L}$
- Magnetic Reynold number $R_m = \frac{V_0 L}{\eta}$

4.1 Mass equation ordering

Using the ordering we obtain

$$\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) =& w_0 n_0 \partial_t \rho + \frac{n_0 V_0}{L} \nabla \cdot (\rho \boldsymbol{u}) \\
=& w_0 n_0 m_i \partial_t n + \frac{n_0 m_i V_0}{L} \nabla \cdot (n \boldsymbol{u}) \\
=& \frac{w_0}{w_{ci}} e B_0 \partial_t n + \frac{m_i V_0}{L} \nabla \cdot (n \boldsymbol{u}) \\
=& \varepsilon \partial_t n + \frac{m_i V_0}{L e B_0} \nabla \cdot (n \boldsymbol{u}) \\
=& \varepsilon \partial_t n + \xi \frac{V_{T_i} m_i}{L e B_0} \nabla \cdot (n \boldsymbol{u}) \\
=& \varepsilon \partial_t n + \xi \frac{V_{T_i}}{L w_{ci}} \nabla \cdot (n \boldsymbol{u})
\end{aligned}$$

Using that $\rho_i^* = \frac{V_{T_i}}{w_{ci}}$ we obtain the following mass equation

Ordering mass equation

$$\varepsilon \partial_t n + \xi \delta \nabla \cdot (n \boldsymbol{u}) = 0$$

4.2 Velocity equation ordering

Using the ordering we obtain

$$\begin{aligned}
EV =&\rho\partial_t \boldsymbol{u} + \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p - \boldsymbol{J}\times\boldsymbol{B} + \nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}\\
=&w_0 n_0 V_0 m_i (n\partial_t \boldsymbol{u}) + \frac{n_0 V_0^2 m_i}{L}(n\boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \frac{p_0}{L}\nabla p - J_0 B_0 \boldsymbol{J}\times\boldsymbol{B} + \frac{\overline{\overline{\boldsymbol{\Pi}}}_0}{L}\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}\\
=&\varepsilon n_0 V_0 w_{ci} m_i (n\partial_t \boldsymbol{u}) + \frac{n_0 V_0^2 m_i}{L}(n\boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \frac{p_0}{L}\nabla p - J_0 B_0 \boldsymbol{J}\times\boldsymbol{B} + \frac{\overline{\overline{\boldsymbol{\Pi}}}_0}{L}\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}\\
=&\varepsilon n_0 V_0 e B_0 (n\partial_t \boldsymbol{u}) + \frac{n_0 V_0^2 m_i}{L}(n\boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \frac{p_0}{L}\nabla p - J_0 B_0 \boldsymbol{J}\times\boldsymbol{B} + \frac{\overline{\overline{\boldsymbol{\Pi}}}_0}{L}\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}\\
=&\varepsilon \xi (V_{T_i} n_0 e B_0)(n\partial_t \boldsymbol{u}) + \frac{n_0 V_0^2 m_i}{L}(n\boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \frac{p_0}{L}\nabla p - J_0 B_0 \boldsymbol{J}\times\boldsymbol{B} + \frac{\overline{\overline{\boldsymbol{\Pi}}}_0}{L}\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}\\
=&\varepsilon \xi (n\partial_t \boldsymbol{u}) + \frac{V_0^2 m_i}{L V_{T_i} e B_0}(n\boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \frac{p_0}{L V_{T_i} n_0 e B_0}\nabla p - \frac{J_0}{V_{T_i} n_0 e}\boldsymbol{J}\times\boldsymbol{B} + \frac{\overline{\overline{\boldsymbol{\Pi}}}_0}{L V_{T_i} n_0 e B_0}\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}
\end{aligned}$$

Since $J_0 = n_0 e V_0$

$$\begin{aligned}
EV =&\varepsilon \xi (n\partial_t \boldsymbol{u}) + \xi^2\delta(n\boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \delta\frac{p_0}{n_0 m_i V_{T_i}^2}\nabla p - \xi\boldsymbol{J}\times\boldsymbol{B} + \frac{\overline{\overline{\boldsymbol{\Pi}}}_0}{L V_{T_i} n_0 e B_0}\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}\\
=&\varepsilon \xi (n\partial_t \boldsymbol{u}) + \xi^2\delta(n\boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \delta\frac{p_0}{n_0 m_i V_{T_i}^2}\nabla p - \xi\boldsymbol{J}\times\boldsymbol{B} + \frac{\overline{\overline{\boldsymbol{\Pi}}}_0}{L V_{T_i} n_0 e B_0}\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}
\end{aligned}$$

We consider that $p_0 = n_0 m_i V_{T_i}^2$ consequently

$$EV = \varepsilon \xi (n\partial_t \boldsymbol{u}) + \xi^2\delta(n\boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \delta\nabla p - \xi\boldsymbol{J}\times\boldsymbol{B} + \frac{\overline{\overline{\boldsymbol{\Pi}}}_0}{L V_{T_i} n_0 e B_0}\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}$$

To finish we use that $\frac{\overline{\overline{\boldsymbol{\Pi}}}_0}{L V_{T_i} n_0 e B_0} = \delta\overline{\overline{\boldsymbol{\Pi}}}_0 \frac{w_{ci}}{V_{T_i}^2 n_0 e B_0} = \delta\frac{\overline{\overline{\boldsymbol{\Pi}}}_0}{p_0}$ and we obtain at the end

Ordering momentum equation

$$\varepsilon \xi (n\partial_t \boldsymbol{u}) + \xi^2\delta(n\boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \delta\nabla p - \xi\boldsymbol{J}\times\boldsymbol{B} + \delta\frac{\overline{\overline{\boldsymbol{\Pi}}}_0}{p_0}\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}} = 0$$

4.3 Pressure equation ordering

Firstly we remark that $\delta\xi = \frac{\rho_i^*}{L}\frac{V_0}{V_{T_i}} = \frac{V_0}{Lw_{ci}}$. We introduce also $q_0 = V_0 p_0$.

$$\begin{aligned}
EP =& \frac{1}{\gamma-1}\partial_t p + \frac{1}{\gamma-1}\boldsymbol{u}\cdot\nabla p + \frac{\gamma}{\gamma-1}p\nabla\cdot\boldsymbol{u} + \nabla\cdot\mathbf{q} \\
&- \frac{1}{\gamma-1}\frac{m_i}{e\rho}\boldsymbol{J}\cdot\left(\nabla p_e - \gamma p_e\frac{\nabla\rho}{\rho}\right) + \overline{\overline{\Pi}} : \nabla\boldsymbol{u} - \overline{\overline{\Pi}}_e : \nabla\left(\frac{m_i}{e\rho}\boldsymbol{J}\right) - \eta|\boldsymbol{J}|^2 \\
=& \frac{1}{\gamma-1}p_0 w_0 \partial_t p + \frac{1}{\gamma-1}\left(\frac{V_0 p_0}{L}\right)\boldsymbol{u}\cdot\nabla p + \frac{\gamma}{\gamma-1}\left(\frac{V_0 p_0}{L}\right)p\nabla\cdot\boldsymbol{u} + \left(\frac{q_0}{L}\right)\nabla\cdot\mathbf{q} \\
&- \frac{1}{\gamma-1}\frac{p_0^e J_0}{Len_0}\frac{\boldsymbol{J}}{n}\cdot\left(\nabla p_e - \gamma p_e\frac{\nabla n}{n}\right) + \frac{\overline{\overline{\Pi}}_0 V_0}{L}\overline{\overline{\Pi}} : \nabla\boldsymbol{u} - (\frac{\overline{\overline{\Pi}}{}_0^e J_0}{Len_0})\overline{\overline{\Pi}}_e : \nabla\left(\frac{1}{n}\boldsymbol{J}\right) - \eta J_0^2|\boldsymbol{J}|^2 \\
=& \frac{1}{\gamma-1}p_0\varepsilon w_{ci}\partial_t p + \frac{1}{\gamma-1}\left(\frac{V_0 p_0}{L}\right)\boldsymbol{u}\cdot\nabla p + \frac{\gamma}{\gamma-1}\left(\frac{V_0 p_0}{L}\right)p\nabla\cdot\boldsymbol{u} + \left(\frac{q_0}{L}\right)\nabla\cdot\mathbf{q} \\
&- \frac{1}{\gamma-1}\frac{p_0^e J_0}{Len_0}\frac{\boldsymbol{J}}{n}\cdot\left(\nabla p_e - \gamma p_e\frac{\nabla n}{n}\right) + \frac{\overline{\overline{\Pi}}_0 V_0}{L}\overline{\overline{\Pi}} : \nabla\boldsymbol{u} - (\frac{\overline{\overline{\Pi}}{}_0^e J_0}{Len_0})\overline{\overline{\Pi}}_e : \nabla\left(\frac{1}{n}\boldsymbol{J}\right) - \eta J_0^2|\boldsymbol{J}|^2 \\
=& \frac{1}{\gamma-1}\varepsilon\partial_t p + \frac{1}{\gamma-1}\xi\eta\boldsymbol{u}\cdot\nabla p + \frac{\gamma}{\gamma-1}\xi\eta p\nabla\cdot\boldsymbol{u} + \left(\frac{q_0}{Lp_0 w_{ci}}\right)\nabla\cdot\mathbf{q} \\
&- \frac{1}{\gamma-1}\frac{p_0^e}{p_0}\frac{J_0}{Len_0 w_{ci}}\frac{\boldsymbol{J}}{n}\cdot\left(\nabla p_e - \gamma p_e\frac{\nabla n}{n}\right) + \frac{\overline{\overline{\Pi}}_0}{p_0}\delta\xi\overline{\overline{\Pi}} : \nabla\boldsymbol{u} - (\frac{\overline{\overline{\Pi}}{}_0^e}{p_0}\frac{J_0}{Len_0 w_{ci}})\overline{\overline{\Pi}}_e : \nabla\left(\frac{1}{n}\boldsymbol{J}\right) - \eta\frac{J_0^2}{w_{ci}p_0}|\boldsymbol{J}|^2 \\
=& \frac{1}{\gamma-1}\varepsilon\partial_t p + \frac{1}{\gamma-1}\delta\xi\boldsymbol{u}\cdot\nabla p + \frac{\gamma}{\gamma-1}\delta\xi p\nabla\cdot\boldsymbol{u} + \left(\frac{q_0\delta}{p_0 v_{T_i}}\right)\nabla\cdot\mathbf{q} \\
&- \frac{1}{\gamma-1}\frac{p_0^e}{p_0}\delta\xi\frac{\boldsymbol{J}}{n}\cdot\left(\nabla p_e - \gamma p_e\frac{\nabla n}{n}\right) + \frac{\overline{\overline{\Pi}}_0}{p_0}\delta\xi\overline{\overline{\Pi}} : \nabla\boldsymbol{u} - \delta\xi\frac{\overline{\overline{\Pi}}{}_0^e}{p_0}\overline{\overline{\Pi}}_e : \nabla\left(\frac{1}{n}\boldsymbol{J}\right) - \eta\frac{J_0^2}{w_{ci}p_0}|\boldsymbol{J}|^2 \\
=& \frac{1}{\gamma-1}\varepsilon\partial_t p + \frac{1}{\gamma-1}\delta\xi\boldsymbol{u}\cdot\nabla p + \frac{\gamma}{\gamma-1}\delta\xi p\nabla\cdot\boldsymbol{u} + \left(\frac{q_0 V_0\delta}{p_0 V_0 V_{T_i}}\right)\nabla\cdot\mathbf{q} \\
&- \frac{1}{\gamma-1}\frac{p_0^e}{p_0}\delta\xi\frac{\boldsymbol{J}}{n}\cdot\left(\nabla p_e - \gamma p_e\frac{\nabla n}{n}\right) + \frac{\overline{\overline{\Pi}}_0}{p_0}\delta\xi\overline{\overline{\Pi}} : \nabla\boldsymbol{u} - \delta\xi\frac{\overline{\overline{\Pi}}{}_0^e}{p_0}\overline{\overline{\Pi}}_e : \nabla\left(\frac{1}{n}\boldsymbol{J}\right) - \eta\frac{B_0}{\nu L}\frac{n_0 e V_0}{m_i n_0 V_{T_i}^2 w_{ci}}|\boldsymbol{J}|^2 \\
=& \frac{1}{\gamma-1}\varepsilon\partial_t p + \frac{1}{\gamma-1}\delta\xi\boldsymbol{u}\cdot\nabla p + \frac{\gamma}{\gamma-1}\delta\xi p\nabla\cdot\boldsymbol{u} + \delta\xi\left(\frac{q_0}{p_0 V_0}\right)\nabla\cdot\mathbf{q} \\
&- \frac{1}{\gamma-1}\frac{p_0^e}{p_0}\delta\xi\frac{\boldsymbol{J}}{n}\cdot\left(\nabla p_e - \gamma p_e\frac{\nabla n}{n}\right) + \frac{\overline{\overline{\Pi}}_0}{p_0}\delta\xi\overline{\overline{\Pi}} : \nabla\boldsymbol{u} - \delta\xi\frac{\overline{\overline{\Pi}}{}_0^e}{p_0}\overline{\overline{\Pi}}_e : \nabla\left(\frac{1}{n}\boldsymbol{J}\right) - \eta\frac{B_0}{\nu L V_0}\frac{e}{m_i w_{ci}}\xi^2|\boldsymbol{J}|^2
\end{aligned}$$

because $\dfrac{J_0}{Ln_0 e w_{ci}} = \delta\xi$ and $J_0 = n_0 e V_0 = \frac{B_0}{\nu L}$. We obtain

Ordering pressure equation

$$\frac{1}{\gamma-1}\varepsilon\partial_t p + \frac{1}{\gamma-1}\delta\xi\boldsymbol{u}\cdot\nabla p + \frac{\gamma}{\gamma-1}\delta\xi p\nabla\cdot\boldsymbol{u} + \delta\xi\nabla\cdot\mathbf{q} - \frac{1}{\gamma-1}\frac{p_0^e}{p_0}\delta\xi\frac{\boldsymbol{J}}{n}\cdot\left(\nabla p_e - \gamma p_e\frac{\nabla n}{n}\right)$$
$$+\frac{\overline{\overline{\Pi}}_0}{p_0}\delta\xi\overline{\overline{\Pi}} : \nabla\boldsymbol{u} - \delta\xi\frac{\overline{\overline{\Pi}}{}_0^e}{p_0}\overline{\overline{\Pi}}_e : \nabla\left(\frac{1}{n}\boldsymbol{J}\right) - \frac{\xi^2}{\mu R_m}|\boldsymbol{J}|^2 = 0$$

4.4 Generalized Ohm Law ordering

After the three classical equation we study the Ohm Law.

$$\begin{aligned}
OL =& \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} - \eta \boldsymbol{J} + \frac{m_i}{\rho e} \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_e + \frac{m_i}{\rho e} \nabla p_e - \frac{m_i}{\rho e} (\boldsymbol{J} \times \boldsymbol{B}) \\
=& \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} - \eta \boldsymbol{J} + \frac{1}{ne} \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_e + \frac{1}{ne} \nabla p_e - \frac{1}{ne} (\boldsymbol{J} \times \boldsymbol{B}) \\
=& E_0 \boldsymbol{E} + V_0 B_0 \boldsymbol{u} \times \boldsymbol{B} - \eta J_0 \boldsymbol{J} + \left(\frac{\overline{\overline{\boldsymbol{\Pi}}}_0^e}{n_0 e} \right) \frac{1}{n} \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_e + \left(\frac{p_0^e}{n_0 e} \right) \frac{1}{n} \nabla p_e - \left(\frac{J_0 B_0}{n_0 e} \right) \frac{1}{n} (\boldsymbol{J} \times \boldsymbol{B}) \\
=& \xi V_{T_i} B_0 \boldsymbol{E} + \xi V_{T_i} B_0 \boldsymbol{u} \times \boldsymbol{B} - \eta J_0 \boldsymbol{J} + \left(\frac{\overline{\overline{\boldsymbol{\Pi}}}_0^e}{L n_0 e} \right) \frac{1}{n} \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_e + \left(\frac{p_0^e}{L n_0 e} \right) \frac{1}{n} \nabla p_e - \left(\frac{J_0 B_0}{n_0 e} \right) \frac{1}{n} (\boldsymbol{J} \times \boldsymbol{B}) \\
=& \xi \boldsymbol{E} + \xi \boldsymbol{u} \times \boldsymbol{B} - \eta \frac{J_0}{V_{T_i} B_0} \boldsymbol{J} + \left(\frac{\overline{\overline{\boldsymbol{\Pi}}}_0^e}{L p_0} \frac{1}{B_0 n_0 e} \frac{p_0}{V_{T_i}} \right) \frac{1}{n} \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_e + \left(\frac{p_0^e}{L p_0} \frac{1}{B_0 n_0 e} \frac{p_0}{V_{T_i}} \right) \frac{1}{n} \nabla p_e - \left(\frac{J_0}{n_0 e V_{T_i}} \right) \frac{1}{n} (\boldsymbol{J} \times \boldsymbol{B}) \\
=& \xi \boldsymbol{E} + \xi \boldsymbol{u} \times \boldsymbol{B} - \eta \frac{J_0}{V_{T_i} B_0} \boldsymbol{J} + \frac{\overline{\overline{\boldsymbol{\Pi}}}_0^e}{p_0} \delta \frac{1}{n} \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_e + \delta \frac{p_0^e}{p_0} \frac{1}{n} \nabla p_e - \xi \frac{1}{n} (\boldsymbol{J} \times \boldsymbol{B})
\end{aligned}$$

To finish we consider the resistive term $\eta \frac{J_0}{V_{T_i} B_0}$. By definition $J_0 = \frac{B_0}{L\mu}$. Using this, we obtain $\eta \dfrac{J_0}{V_{T_i} B_0} = \xi \dfrac{\eta}{V_0 L} = \xi \dfrac{1}{R_m}$.

Ordering generalized Ohm Low

$$\xi \boldsymbol{E} + \xi \boldsymbol{u} \times \boldsymbol{B} - \frac{\xi}{\mu R_m} \boldsymbol{J} + \frac{\overline{\overline{\boldsymbol{\Pi}}}_0^e}{p_0} \delta \frac{1}{n} \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_e + \delta \frac{p_0^e}{p_0} \frac{1}{n} \nabla p_e - \xi \frac{1}{n} (\boldsymbol{J} \times \boldsymbol{B}) = 0$$

To find some simplifications we propose to study the ordering in the stress tensor and in the velocity field.

4.5 Stress tensor ordering

To finish we consider the ordering for the stress tensor. We consider the stress tensor which can be write on the following form

$$\Pi = \left(\frac{1}{\nu} \overline{\overline{\boldsymbol{\Pi}}}_{\parallel}^s + \frac{1}{w_{ci}} \overline{\overline{\boldsymbol{\Pi}}}_{cross}^s + \frac{\nu}{w_{ci}^2} \overline{\overline{\boldsymbol{\Pi}}}_{\perp}^s \right)$$

The normalization of the strain tensor is given by $\frac{V_0}{L}$. We obtain

$$\overline{\overline{\mathbf{\Pi}}}_0\overline{\overline{\mathbf{\Pi}}} = \left(\frac{p_0 p V_0}{L\nu}\overline{\overline{\mathbf{\Pi}}}^s_\parallel + \frac{p_0 V_0 p}{L w_{ci}}\overline{\overline{\mathbf{\Pi}}}^s_{cross} + \frac{p_0 V_0 p \nu}{L w_{ci}^2}\overline{\overline{\mathbf{\Pi}}}^s_\perp\right)$$

$$\overline{\overline{\mathbf{\Pi}}}_0\overline{\overline{\mathbf{\Pi}}} = \delta\xi\frac{L w_{ci}}{V_0}\left(\frac{p_0 p V_0}{L\nu}\overline{\overline{\mathbf{\Pi}}}^s_\parallel + \frac{p_0 V_0 p}{L w_{ci}}\overline{\overline{\mathbf{\Pi}}}^s_{cross} + \frac{p_0 V_0 p \nu}{L w_{ci}^2}\Pi^s_\perp\right)$$

$$\overline{\overline{\mathbf{\Pi}}}_0\overline{\overline{\mathbf{\Pi}}} = \delta\xi p_0\left(\frac{w_{ci}}{\nu}\overline{\overline{\mathbf{\Pi}}}^s_\parallel + \overline{\overline{\mathbf{\Pi}}}^s_{cross} + \frac{\nu}{w_{ci}}\overline{\overline{\mathbf{\Pi}}}^s_\perp\right)$$

Ordering stress tensor

$$\overline{\overline{\mathbf{\Pi}}}_0\overline{\overline{\mathbf{\Pi}}} = \delta\xi p_0\left(\frac{w_{ci}}{\nu}\overline{\overline{\mathbf{\Pi}}}^s_\parallel + \overline{\overline{\mathbf{\Pi}}}^s_{cross} + \frac{\nu}{w_{ci}}\overline{\overline{\mathbf{\Pi}}}^s_\perp\right)$$

To finish we consider the ordering to the velocity to simplify the full velocity fields.

4.6 Velocity ordering

Firstly we take the ion velocity equation obtained with the ion momentum equation and ion continuity equation without particle source:

$$m_i n_i \partial_t \boldsymbol{u}_i + m_i n_i \boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i + \nabla p_i = e n_i(\mathbf{E} + \boldsymbol{u}_i \times \boldsymbol{B}) - \nabla \cdot \overline{\overline{\mathbf{\Pi}}}_i + \boldsymbol{R}_i$$

We obtain

$$\boldsymbol{u}_i \times \boldsymbol{B} = -\boldsymbol{E} + \frac{m_i}{e}(\partial_t \boldsymbol{u}_i + \boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i) + \frac{1}{n_i e}\nabla p_i + \frac{1}{n_i e}\nabla \cdot \overline{\overline{\mathbf{\Pi}}}_i - \frac{1}{n_i e}\boldsymbol{R}_i$$

Now we apply the operator $\boldsymbol{B}\times$. We obtain

$$\frac{\boldsymbol{B} \times \boldsymbol{u}_i \times \boldsymbol{B}}{|\boldsymbol{B}|^2} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{|\boldsymbol{B}|^2} + \frac{m_i}{e|\boldsymbol{B}|^2}\boldsymbol{B} \times (\partial_t \boldsymbol{u}_i + \boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i) + \frac{\boldsymbol{B}}{n_i e|\boldsymbol{B}|^2} \times (\nabla p_i + \nabla \cdot \overline{\overline{\mathbf{\Pi}}}_i - \boldsymbol{R}_i)$$

which gives

$$\boldsymbol{u}_i = (\boldsymbol{u}_i \cdot \boldsymbol{B})\frac{\boldsymbol{B}}{|\boldsymbol{B}|^2} + \frac{\boldsymbol{E} \times \boldsymbol{B}}{|\boldsymbol{B}|^2} + \frac{m_i}{e|\boldsymbol{B}|^2}\boldsymbol{B} \times (\partial_t \boldsymbol{u}_i + \boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i) + \frac{\boldsymbol{B}}{n_i e|\boldsymbol{B}|^2} \times (\nabla p_i + \nabla \cdot \overline{\overline{\mathbf{\Pi}}}_i - \boldsymbol{R}_i)$$

We remark that the ion velocity can be write as a sum of some drift

$$\boldsymbol{u}_i = \boldsymbol{u}_\parallel + \boldsymbol{u}_E + \boldsymbol{u}_{*i} + \boldsymbol{u}_{pi} + \boldsymbol{u}_{\Pi_i} + \boldsymbol{u}_{R_i}$$

with the different "drifts velocities":

- **Parallel velocity**: $\dfrac{m_i}{e|\boldsymbol{B}|^2}\boldsymbol{B} \times (\partial_t \boldsymbol{u}_i + \boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i)$
- **Classical MHD perpendicular velocity** : $\dfrac{\boldsymbol{E} \times \boldsymbol{B}}{|\boldsymbol{B}|^2}$

- **Ion perpendicular polarization drift**: $\dfrac{m_i}{e|\boldsymbol{B}|^2}\boldsymbol{B} \times (\partial_t \boldsymbol{u}_i + \boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i)$
- **Diamagnetic perpendicular drift**: $\dfrac{\boldsymbol{B}}{n_i e|\boldsymbol{B}|^2} \times \nabla p_i$
- **Stress perpendicular drift**: $\dfrac{\boldsymbol{B}}{n_i e|\boldsymbol{B}|^2} \times \nabla \cdot \overline{\overline{\overline{\boldsymbol{\Pi}}}}_i$
- **friction perpendicular drift**: $-\dfrac{\boldsymbol{B}}{n_i e|\boldsymbol{B}|^2} \times \boldsymbol{R}_i$

Now we apply the ordering on this decomposition

$$
\begin{aligned}
\boldsymbol{u}_i &= (\boldsymbol{u}_i \cdot \boldsymbol{B})\frac{\boldsymbol{B}}{|\boldsymbol{B}|^2} + \frac{\boldsymbol{E} \times \boldsymbol{B}}{|\boldsymbol{B}|^2} + \frac{m_i}{e}\frac{\boldsymbol{B}}{|\boldsymbol{B}|^2} \times (\partial_t \boldsymbol{u}_i + \boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i) + \frac{\boldsymbol{B}}{n_i e|\boldsymbol{B}|^2} \times (\nabla p_i + \nabla \cdot \overline{\overline{\overline{\boldsymbol{\Pi}}}}_i - \boldsymbol{R}_i) \\
V_0 \boldsymbol{u}_i &= V_0(\boldsymbol{u}_i \cdot \boldsymbol{B})\frac{\boldsymbol{B}}{|\boldsymbol{B}|^2} + V_0\frac{\boldsymbol{E} \times \boldsymbol{B}}{|\boldsymbol{B}|^2} + \frac{m_i V_0}{eB_0} \times \left(w_0 \partial_t \boldsymbol{u}_i + \frac{V_0}{L}\boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i\right) \\
&\quad + \frac{\boldsymbol{B}}{n|\boldsymbol{B}|^2} \times \left(\frac{p_0}{LB_0 e n_0}\nabla p_i + \frac{\overline{\overline{\overline{\boldsymbol{\Pi}}}}_0}{LB_0 e n_0}\nabla \cdot \overline{\overline{\overline{\boldsymbol{\Pi}}}}_i - \frac{R_0}{B_0 e n_0}\boldsymbol{R}_i\right) \\
\xi \boldsymbol{u}_i &= \xi(\boldsymbol{u}_i \cdot \boldsymbol{B})\frac{\boldsymbol{B}}{|\boldsymbol{B}|^2} + \xi\frac{\boldsymbol{E} \times \boldsymbol{B}}{|\boldsymbol{B}|^2} + \frac{m_i V_0}{eB_0 V_{T_i}} \times \left(w_0 \partial_t \boldsymbol{u}_i + \frac{V_0}{L}\boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i\right) \\
&\quad + \frac{\boldsymbol{B}}{n|\boldsymbol{B}|^2} \times \left(\frac{p_0}{LB_0 e n_0 V_{T_i}}\nabla p_i + \frac{\overline{\overline{\overline{\boldsymbol{\Pi}}}}_0}{LB_0 e n_0 V_{T_i}}\nabla \cdot \overline{\overline{\overline{\boldsymbol{\Pi}}}}_i - \frac{R_0}{B_0 e n_0 V_{T_i}}\boldsymbol{R}_i\right) \\
\xi \boldsymbol{u}_i &= \xi(\boldsymbol{u}_i \cdot \boldsymbol{B})\frac{\boldsymbol{B}}{|\boldsymbol{B}|^2} + \xi\frac{\boldsymbol{E} \times \boldsymbol{B}}{|\boldsymbol{B}|^2} + \varepsilon\xi\partial_t \boldsymbol{u}_i + \xi^2 \delta \boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i \\
&\quad + \frac{\boldsymbol{B}}{n|\boldsymbol{B}|^2} \times \left(\delta \nabla p_i + \frac{\overline{\overline{\overline{\boldsymbol{\Pi}}}}_0}{p_0}\delta \nabla \cdot \overline{\overline{\overline{\boldsymbol{\Pi}}}}_i - \frac{R_0}{B_0 e n_0 V_{T_i}}\boldsymbol{R}_i\right)
\end{aligned}
$$

Therefore we obtain the following decomposition of the velocity

$$
\xi \boldsymbol{u}_i = \xi(\boldsymbol{u}_i \cdot \boldsymbol{B})\frac{\boldsymbol{B}}{\mid \boldsymbol{B} \mid^2} + \xi\frac{\boldsymbol{E} \times \boldsymbol{B}}{|\boldsymbol{B}|^2} + \varepsilon\xi\partial_t \boldsymbol{u}_i + \xi^2 \delta \boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i + \frac{\boldsymbol{B}}{n|\boldsymbol{B}|^2} \times \left(\delta \nabla p_i + \frac{\overline{\overline{\overline{\boldsymbol{\Pi}}}}_0}{p_0}\delta \nabla \cdot \overline{\overline{\overline{\boldsymbol{\Pi}}}}_i - \frac{R_0}{B_0 e n_0 V_{T_i}}\boldsymbol{R}_i\right)
$$

using $R_0 = \eta e n_0 J_0$ we obtain

Ordering velocity

$$
\begin{aligned}
\xi \boldsymbol{u} &= \xi(\boldsymbol{u}_i \cdot \boldsymbol{B})\frac{\boldsymbol{B}}{\mid \boldsymbol{B} \mid^2} + \xi\frac{\boldsymbol{E} \times \boldsymbol{B}}{|\boldsymbol{B}|^2} + \varepsilon\xi\partial_t + \xi^2 \delta \boldsymbol{u}_i \boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i \\
&\quad + \frac{\boldsymbol{B}}{n \mid \boldsymbol{B} \mid^2} \times \left(\delta \nabla p_i + \frac{\overline{\overline{\overline{\boldsymbol{\Pi}}}}_0}{p_0}\delta \nabla \cdot \overline{\overline{\overline{\boldsymbol{\Pi}}}}_i - \frac{\xi}{R_m}\boldsymbol{R}_i\right)
\end{aligned}
$$

4.7 Tokamak Regime and model associated

Assumption regime

- **Assumption I**: $\frac{\nu}{w_{ci}} \approx \delta^2$
- **Assumption II**: $\xi \approx \delta$
- **Assumption III**: $\varepsilon \approx \delta^2$
- **Assumption IV** $\frac{|\overline{\overline{\Pi}}_e|}{|\overline{\overline{\Pi}}_i|} \approx \frac{m_e}{m_i} << 1$

At the end the Ordering for the extended MHD is given by

$$\begin{cases} \delta^2 \partial_t n + \xi \delta^2 \cdot (n\boldsymbol{u}) = 0 \\ \\ \delta^3 (n \partial_t \boldsymbol{u}) + \delta^3 (n\boldsymbol{u} \cdot \nabla \boldsymbol{u}) + \delta \nabla p - \delta \boldsymbol{J} \times \boldsymbol{B} \delta \overline{\overline{\boldsymbol{\Pi}}}_{\parallel} + \delta^3 \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{gv} + \delta^5 \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{\perp} = 0 \\ \\ \frac{1}{\gamma-1}\delta^2 \partial_t p + \frac{1}{\gamma-1}\delta^2 \boldsymbol{u} \cdot \nabla p + \frac{\gamma}{\gamma-1}\delta^2 p \nabla \cdot \boldsymbol{u} + \delta^2 \nabla \cdot \mathbf{q} = \frac{1}{\gamma-1}\frac{p_0^e}{p_0}\delta^2 \frac{\boldsymbol{J}}{n} \cdot \left(\nabla p_e - \gamma p_e \frac{\nabla n}{n} \right) \\ -\delta^2 \overline{\overline{\boldsymbol{\Pi}}}_{\parallel} : \nabla \boldsymbol{u} - \delta^4 \overline{\overline{\boldsymbol{\Pi}}}_{gv} : \nabla \boldsymbol{u} - \delta^6 \overline{\overline{\boldsymbol{\Pi}}}_{\perp} : \nabla \boldsymbol{u} + \delta^2 \frac{\overline{\overline{\boldsymbol{\Pi}}}_0^e}{p_0} \Pi_e : \nabla \left(\frac{1}{n} \boldsymbol{J} \right) + \frac{\delta^2}{\mu R_m} |\boldsymbol{J}|^2 \\ \\ \delta^2 \partial_t \boldsymbol{B} = \delta^2 \nabla \times \boldsymbol{E} \\ \\ \delta \boldsymbol{E} + \delta \boldsymbol{u} \times \boldsymbol{B} - \frac{\xi}{\mu R_m} \boldsymbol{J} + \frac{\overline{\overline{\boldsymbol{\Pi}}}_0^e}{p_0} \delta \frac{1}{n} \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_e + \delta \frac{p_0^e}{p_0} \frac{1}{n} \nabla p_e - \delta \frac{1}{n} (\boldsymbol{J} \times \boldsymbol{B}) = 0 \\ \\ \delta \nabla \times \boldsymbol{B} = \boldsymbol{J} \\ \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

We neglected the term homogeneous to $O(\delta^4)$, we compute the equation with dimensionless variables and we coupled with previous velocity decomposition the model with two pressure. Since the effect of electronic tensor a small comparing the the effect of ion stress tensor we neglect the electronic stress tensor effect

Full MHD model in Tokamak regime

$$
\begin{cases}
\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0 \\
\rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \boldsymbol{u} + \nabla p = \boldsymbol{J} \times \boldsymbol{B} + - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{gv} - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{\parallel} \\
\partial_t \frac{1}{\gamma - 1} p_i + \frac{1}{\gamma - 1} \boldsymbol{u} \cdot \nabla p_i + \frac{\gamma}{\gamma - 1} p_i \nabla \cdot \boldsymbol{u} + \nabla \cdot \mathbf{q}_i + \overline{\overline{\boldsymbol{\Pi}}}_{\parallel} : \nabla \boldsymbol{u} + \overline{\overline{\boldsymbol{\Pi}}}_{gv} : \nabla \boldsymbol{u} \\
= 3 \frac{\rho_e}{\tau_e m_i} (T_i - T_e) \\
\partial_t \frac{1}{\gamma - 1} p_e + \frac{1}{\gamma - 1} \boldsymbol{u} \cdot \nabla p_e + \frac{\gamma}{\gamma - 1} p_e \nabla \cdot \boldsymbol{u} + \nabla \cdot \mathbf{q}_e \\
= \frac{1}{\gamma - 1} \frac{m_i}{\rho e} \boldsymbol{J} \cdot \left(\nabla p_e - \gamma p_e \frac{\nabla \rho}{\rho} \right) - 3 \frac{\rho_e}{\tau_e m_i} (T_i - T_e) + \eta |\boldsymbol{J}|^2 \\
\partial_t \boldsymbol{B} = - \nabla \times \boldsymbol{E} \\
\boldsymbol{E} = \left(- \boldsymbol{u} \times \boldsymbol{B} + \eta \boldsymbol{J} - \frac{m_i}{\rho e} \nabla p_e + \frac{m_i}{\rho e} (\boldsymbol{J} \times \boldsymbol{B}) \right) \\
\mu_0 \nabla \times \boldsymbol{B} = \boldsymbol{J}, \quad \nabla \cdot \boldsymbol{B} = 0 \\
\boldsymbol{u} = \boldsymbol{u}_E + \boldsymbol{u}_{\parallel} + \boldsymbol{u}_i^*, \quad \boldsymbol{u}_i^* = \frac{m_i}{e \rho} \frac{\boldsymbol{B} \times \nabla p_i}{| \boldsymbol{B} |^2} \text{ and } \boldsymbol{u}_E = \frac{\boldsymbol{E} \times \boldsymbol{B}}{| \boldsymbol{B} |^2}
\end{cases}
\tag{4.17}
$$

5 Full MHD proposed for JOREK

In this section we propose to simplify a little bit the previous model (4.17) simplifying the stress tensor and splitting the perpendicular to the parallel velocity. For the new model obtained we proposed to check the energy conservation.

5.1 Simplify stress tensor and energy conservation

To simplify we note

- $\overline{v} = \frac{\boldsymbol{B}}{|\boldsymbol{B}|}$
- $\nabla_{\parallel} = (\mathbf{b} \cdot \nabla) \mathbf{b}$
- $\nabla_{\perp} = \nabla - \nabla_{\parallel}$

Now we consider the the parallel Tensor. Using the paper of Zeiler (need to by justify) we assume that

$$
\nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{\parallel} = G \mathbf{b} \cdot \nabla \mathbf{b} - \frac{1}{3} \nabla G + \parallel \boldsymbol{B} \parallel \nabla_{\parallel} \left(\frac{G}{\parallel \boldsymbol{B} \parallel} \right)
$$

and

$$
G = -3 \eta_0 \left(\mathbf{b} \nabla v_{\parallel} - (\mathbf{b} \cdot \nabla \mathbf{b}) \cdot \boldsymbol{u} - \frac{1}{3} \nabla \cdot \boldsymbol{u} \right)
$$

We assume that the model of $\boldsymbol{B}$ is constant or close to constant in space. Consequently

$$\nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{\parallel} \approx G\mathbf{b} \cdot \nabla \mathbf{b} - \frac{1}{3}\nabla G + \nabla_{\parallel} G$$

Now we propose to simplify G.

$$G = -3\eta_0(\mathbf{b} \cdot \nabla u v_{\parallel} - ((\mathbf{b} \cdot \nabla \mathbf{b}) \cdot \boldsymbol{u}_{\perp}) - \underbrace{((\mathbf{b} \cdot \nabla \mathbf{b}) \cdot \boldsymbol{u}_{\parallel})}_{=0} - \frac{1}{3}\nabla \cdot \boldsymbol{u}_{\perp} - \frac{1}{3}\nabla \cdot \boldsymbol{u}_{\parallel})$$

Using $\nabla \cdot \boldsymbol{u}_{\parallel} = \mathbf{b} \cdot \nabla v_{\parallel}$. We obtain at the end

$$G = -\eta_0(2\mathbf{b} \cdot \nabla v_{\parallel} - 3((\mathbf{b} \cdot \nabla \mathbf{b}) \cdot \boldsymbol{u}_{\perp}) - \nabla \cdot \boldsymbol{u}_{\perp})$$

To finish we assume that $\overline{\overline{\boldsymbol{\Pi}}}_{\parallel} : \nabla \boldsymbol{u} = -\frac{1}{3\eta_0} G^2$. Now we consider the gyro-viscous tensor and take an approximation called FLR (finite Larmor radius).

$$\nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{gv} = -\rho \boldsymbol{u}_i^* \cdot \nabla \boldsymbol{u} + p_i \left(\nabla \times \frac{m_i \mathbf{b}}{e \parallel \boldsymbol{B} \parallel} \right) \cdot \nabla \boldsymbol{u} + \nabla_{\perp} \left(\frac{m_i p_i}{2e \parallel \boldsymbol{B} \parallel} \nabla \cdot \mathbf{b} \times \boldsymbol{u} \right) + \mathbf{b} \times \nabla \left(\frac{m_i p_i}{2e \parallel \boldsymbol{B} \parallel} \nabla_{\perp} \cdot \boldsymbol{u} \right)$$

It is assume that the heating $\overline{\overline{\boldsymbol{\Pi}}}_{gv} : \nabla \boldsymbol{u} = 0$. At the end we consider the following model

Bi-pressure MHD for tokamak with simplify tensors

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0 \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \boldsymbol{J} \times \boldsymbol{B} - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{\parallel} - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{gv} \\ \partial_t \frac{1}{\gamma - 1} p_i + \frac{1}{\gamma - 1} \boldsymbol{u} \cdot \nabla p_i + \frac{\gamma}{\gamma - 1} p_i \nabla \cdot \boldsymbol{u} + \nabla \cdot \mathbf{q}_i + \overline{\overline{\boldsymbol{\Pi}}}_{\parallel} : \nabla \boldsymbol{u} + \overline{\overline{\boldsymbol{\Pi}}}_{gv} : \nabla \boldsymbol{u} \\ = 3 \frac{\rho_e}{\tau_e m_i} (T_i - T_e) \\ \partial_t \frac{1}{\gamma - 1} p_e + \frac{1}{\gamma - 1} \boldsymbol{u} \cdot \nabla p_e + \frac{\gamma}{\gamma - 1} p_e \nabla \cdot \boldsymbol{u} + \nabla \cdot \mathbf{q}_e \\ = \frac{1}{\gamma - 1} \frac{m_i}{\rho e} \boldsymbol{J} \cdot \left(\nabla p_e - \gamma p_e \frac{\nabla \rho}{\rho} \right) - 3 \frac{\rho_e}{\tau_e m_i} (T_i - T_e) + \eta |\boldsymbol{J}|^2 \\ \partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E} \\ \boldsymbol{E} = \left(-\boldsymbol{u} \times \boldsymbol{B} + \eta \boldsymbol{J} - \frac{m_i}{\rho e} \nabla p_e + \frac{m_i}{\rho e} (\boldsymbol{J} \times \boldsymbol{B}) \right) \\ \mu_0 \nabla \times \boldsymbol{B} = \boldsymbol{J}, \quad \nabla \cdot \boldsymbol{B} = 0 \end{cases} \tag{5.18}$$

with

$$\begin{cases} \boldsymbol{u} = \boldsymbol{u}_\perp + \boldsymbol{u}_\parallel, \quad \boldsymbol{u}_\perp = \boldsymbol{u}_E + \boldsymbol{u}_i^*, \quad \boldsymbol{u}_i^* = \frac{m_i}{e\rho} \frac{\boldsymbol{B} \times \nabla p_i}{|\boldsymbol{B}|^2}, \quad \boldsymbol{u}_E = \frac{\boldsymbol{E} \times \boldsymbol{B}}{|\boldsymbol{B}|^2}, \quad \boldsymbol{u}_\parallel = v_\parallel \frac{\boldsymbol{B}}{|\boldsymbol{B}|} \\ \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{gv} = -\rho \boldsymbol{u}_i^* \cdot \nabla \boldsymbol{u} + p_i \left(\nabla \times \frac{m_i \mathbf{b}}{e \| \boldsymbol{B} \|} \right) \cdot \nabla \boldsymbol{u} \\ + \nabla_\perp \left(\frac{m_i p_i}{2e \| \boldsymbol{B} \|} \nabla \cdot \mathbf{b} \times \boldsymbol{u} \right) + \mathbf{b} \times \nabla \left(\frac{m_i p_i}{2e \| \boldsymbol{B} \|} \nabla_\perp \cdot \boldsymbol{u} \right) \\ \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{\parallel} = G \mathbf{b} \cdot \nabla \mathbf{b} - \frac{1}{3} \nabla G + \nabla_\parallel G, \quad \overline{\overline{\boldsymbol{\Pi}}}_{\parallel} : \nabla \boldsymbol{u} = -\frac{1}{3\eta_0} G^2 \\ G = -\eta_0 (2 \mathbf{b} \cdot \nabla v_\parallel - 3((\mathbf{b} \cdot \nabla \mathbf{b}) \cdot \boldsymbol{u}_\perp) - \nabla \cdot \boldsymbol{u}_\perp) \end{cases} \tag{5.19}$$

Lemma

We assume that all the quantity are equate to zero at the boundary. The model (**??**)-(5.19) conserve the total energy of the MHD:

$$\partial_t \int_\Omega \left(\frac{\| \boldsymbol{B} \|^2}{2} + \rho \frac{\| \boldsymbol{u} \|^2}{2} + \frac{1}{\gamma - 1}(p_e + p_i) \right) = 0$$

Proof. Firstly we introduce the equation of the total pressure equation $p = p_e + p_e$. We obtain

$$\partial_t \frac{1}{\gamma-1} p + \frac{1}{\gamma-1} \boldsymbol{u} \cdot \nabla p + \frac{\gamma}{\gamma-1} p \nabla \cdot \boldsymbol{u} + \nabla \cdot (\mathbf{q}_i + \mathbf{q}_e) + \overline{\overline{\boldsymbol{\Pi}}}_\parallel : \nabla \boldsymbol{u} + \overline{\overline{\boldsymbol{\Pi}}}_{gv} : \nabla \boldsymbol{u} = -\frac{1}{\gamma-1} \frac{m_i}{\rho e} \boldsymbol{J} \cdot \left(\nabla p_e - \gamma p_e \frac{\nabla \rho}{\rho} \right)$$

Now Multiply the first equation by $\frac{|\boldsymbol{u}|^2}{2}$ the second by $\boldsymbol{u}$ and summing the two equation we obtain

$$\partial_t \left(\rho \frac{| \boldsymbol{u} |^2}{2} \right) + \nabla \cdot \left(\rho \frac{| \boldsymbol{u} |^2}{2} \boldsymbol{u} \right) + \boldsymbol{u} \cdot \nabla p = \boldsymbol{u} \cdot (\boldsymbol{J} \times \boldsymbol{B}) - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_\parallel \cdot \boldsymbol{u} - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{gv} \cdot \boldsymbol{u} \tag{5.20}$$

Mutipliying by $\boldsymbol{B}$ the magnetic equation we obtain

$$\partial_t \left(\frac{| \boldsymbol{B} |^2}{2} \right) = \boldsymbol{B} \cdot \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - \eta \boldsymbol{B} \cdot \nabla \times \boldsymbol{J} - \boldsymbol{B} \cdot \nabla \times \frac{m_i}{\rho e} \nabla p_e + \boldsymbol{B} \cdot \nabla \times \frac{m_i}{\rho e} (\boldsymbol{J} \times \boldsymbol{B})$$

We integrate on the domain and integrate by part all the term and the right and use $\boldsymbol{J} = \nabla \times \boldsymbol{B}$ to obtain

$$\partial_t \int_\Omega \left(\frac{| \boldsymbol{B} |^2}{2} \right) = \int_\Omega (\boldsymbol{u} \times \boldsymbol{B}) \cdot \boldsymbol{J} - \int_\Omega \eta \mid \boldsymbol{J} \mid^2 - \int_\Omega \boldsymbol{J} \cdot \frac{m_i}{\rho e} \nabla p_e + \underbrace{\int_\omega \boldsymbol{J} \cdot \frac{m_i}{\rho e} (\boldsymbol{J} \times \boldsymbol{B})}_{=0}$$

Integration the kinetic equation and summing with the magnetic equation we obtain

$$\partial_t \int_\Omega \left(\rho \frac{| \boldsymbol{u} |^2}{2} \right) + \partial_t \int_\Omega \left(\frac{| \boldsymbol{B} |^2}{2} \right) = \underbrace{\int_\Omega \nabla \cdot \left(\rho \frac{| \boldsymbol{u} |^2}{2} \boldsymbol{u} \right)}_{=0} + \int_\Omega \boldsymbol{u} \cdot \nabla p - \int_\Omega \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_\parallel \cdot \boldsymbol{u} - \int_\Omega \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{gv} \cdot \boldsymbol{u}$$

$$+ \underbrace{\int_\Omega (\boldsymbol{u} \times \boldsymbol{B}) \cdot \boldsymbol{J} + \int_\Omega \boldsymbol{u} \cdot (\boldsymbol{J} \times \boldsymbol{B})}_{=0} - \int_\Omega \eta \mid \boldsymbol{J} \mid^2 - \int_\Omega \boldsymbol{J} \cdot \frac{m_i}{\rho e} \nabla p_e$$

To kill the first rem we use the flue-divergence theorem and the boundary condition. for the second we use algebraic property of the scalar product and vectorial product. Now we integrate the pressure equation and we add this equation to the previous one. We use the same computation that before we obtain

$$\partial_t \int_\Omega \left(\rho \frac{| \boldsymbol{u} |^2}{2} + \frac{| \boldsymbol{B} |^2}{2} + \frac{1}{\gamma-1}(p_e + p_i) \right) = \underbrace{\frac{\gamma}{\gamma-1} \int_\Omega \nabla \cdot (p \boldsymbol{u})}_{=0} - \int_\Omega \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_\parallel \cdot \boldsymbol{u} - \int_\Omega \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{gv} \cdot \boldsymbol{u} + \underbrace{\int_\Omega \nabla \cdot \mathbf{q}}_{=0}$$

$$- \int_\Omega \overline{\overline{\boldsymbol{\Pi}}}_\parallel : \boldsymbol{u} - \int_\Omega \overline{\overline{\boldsymbol{\Pi}}}_{gv} : \boldsymbol{u} \underbrace{- \int_\Omega \frac{1}{\gamma-1} \frac{m_i}{\rho e} \boldsymbol{J} \cdot \left(\nabla p_e - \gamma p_e \frac{\nabla \rho}{\rho} \right) - \int_\Omega \boldsymbol{J} \cdot \frac{m_i}{\rho e} \nabla p_e}_{=0}$$

The Viscous heating associated with the gyro-viscous tensor is equal to zero (previous assumption need to be justify). Now we consider the term associated with the parallel tensor. By assumption we have

$$\overline{\overline{\boldsymbol{\Pi}}}_{\|} : \boldsymbol{u} = -\frac{1}{3\eta_0} G^2$$

Now we consider

$$\int_{\Omega} \nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}_{\|}\cdot\boldsymbol{u} = \underbrace{\int_{\Omega}\left(G(\mathbf{b}\cdot\nabla\mathbf{b}) - \frac{1}{3}\nabla G\right)\cdot\boldsymbol{u}_{\perp}}_{(1)} + \underbrace{\int_{\Omega}\left(\nabla_{\|}G\right)\cdot\boldsymbol{u}_{\perp}}_{(2)} + \underbrace{\int_{\Omega}\left(G(\mathbf{b}\cdot\nabla\mathbf{b}) - \frac{1}{3}\nabla G + \nabla_{\|}\right)\cdot\boldsymbol{u}_{\|}}_{(3)} \tag{5.21}$$

The term $\int_{\Omega}\left(\nabla_{\|}G\right)\cdot\boldsymbol{u}_{\perp} = \int_{\Omega}(\mathbf{b}\cdot\nabla G)\mathbf{b}\cdot\boldsymbol{u}_{\perp} = 0$. Now we compute the third term called (3)

$$\begin{aligned}
(3) &= \int_{\Omega}\left(G(\mathbf{b}\cdot\nabla\mathbf{b}) - \frac{1}{3}\nabla G + \nabla_{\|}G\right)\cdot\boldsymbol{u}_{\|}\\
&= \int_{\Omega}\underbrace{G\mathbf{b}\cdot\nabla\mathbf{b}\cdot\boldsymbol{u}_{\|}}_{=0} + \int_{\Omega}\left(-\frac{1}{3}\nabla G + \nabla_{\|}G\right)\cdot\boldsymbol{u}_{\|}\\
&= -\int_{\Omega}\frac{1}{3}\nabla G\cdot\boldsymbol{u}_{\|} + \int_{\Omega}\nabla_{\|}G\cdot\boldsymbol{u}_{\|}\\
&= \int_{\Omega}\frac{1}{3}G\nabla\cdot\boldsymbol{u}_{\|} + \int_{\Omega}\nabla_{\|}G\cdot\boldsymbol{u}_{\|}\\
&= \int_{\Omega}\frac{1}{3}G(v_{\|}\nabla\cdot\mathbf{b} + \mathbf{b}\cdot\nabla v_{\|}) + \int_{\Omega}\nabla_{\|}G\cdot\boldsymbol{u}_{\|}\\
&= \int_{\Omega}\frac{1}{3}G\mathbf{b}\cdot\nabla v_{\|} + \int_{\Omega}\nabla_{\|}G\cdot\boldsymbol{u}_{\|}\\
&= \int_{\Omega}\frac{1}{3}G\mathbf{b}\cdot\nabla v_{\|} + \int_{\Omega}\mathbf{b}\cdot\nabla G \mid \mathbf{b}\mid^2 v_{\|}\\
&= \int_{\Omega}\frac{1}{3}G\mathbf{b}\cdot\nabla v_{\|} - \int_{\Omega}G\mathbf{b}\cdot\nabla(\mid \mathbf{b}\mid^2 v_{\|})\\
&= \int_{\Omega}\frac{1}{3}G\mathbf{b}\cdot\nabla v_{\|} - \int_{\Omega}G\mathbf{b}\cdot\nabla u v_{\|}\\
&= -\frac{2}{3}\int_{\Omega}G\mathbf{b}\cdot\nabla v_{\|}
\end{aligned}$$

■

By definition of G we have

$$\frac{G}{3\eta_0} = \left(-\frac{2}{3}\mathbf{b}\cdot\nabla v_{\|} + (\mathbf{b}\cdot\nabla\mathbf{b})\cdot\boldsymbol{u} + \frac{1}{3}\nabla\cdot\boldsymbol{u}_{\perp}\right) = \left(-\frac{2}{3}\mathbf{b}\cdot\nabla u_{\|} + (\mathbf{b}\cdot\nabla\mathbf{b})\cdot\boldsymbol{u}_{\perp} + \frac{1}{3}\nabla\cdot\boldsymbol{u}_{\perp}\right)$$

$$\begin{aligned}
(1) &= \int_{\Omega}\left(G(\mathbf{b}\cdot\nabla\mathbf{b}) - \frac{1}{3}\nabla G\right)\cdot\boldsymbol{u}_{\perp}\\
&= \int_{\Omega}G(\mathbf{b}\cdot\nabla\mathbf{b})\cdot\boldsymbol{u}_{\perp} + \frac{1}{3}\int_{\Omega}G\nabla\cdot\boldsymbol{u}_{\perp}\\
&= \frac{1}{3\eta_0}\int_{\Omega}G^2 + \frac{2}{3}\int_{\Omega}G\mathbf{b}\cdot\nabla v_{\|}
\end{aligned}$$

At the end
$$\int_\Omega \nabla\cdot\overline{\overline{\mathbf{\Pi}}}_\| \cdot \boldsymbol{u} = \frac{1}{3\eta_0}\int_\Omega G^2$$
Consequently the term $\nabla\cdot\overline{\overline{\mathbf{\Pi}}}_\|\cdot\boldsymbol{u}$ dissipate the energy and the viscous heating associated balanced exactly the dissipation. Now we add

$$\partial_t \int_\Omega \left(\rho\frac{|\,\boldsymbol{u}\,|^2}{2} + \frac{|\,\boldsymbol{B}\,|^2}{2} + \frac{1}{\gamma-1}(p_e+p_i)\right) = -\int_\Omega \nabla\cdot\overline{\overline{\mathbf{\Pi}}}_{gv}\cdot\boldsymbol{u}$$

Now we note
$$(1) = -\int_\Omega (\rho\boldsymbol{u}_i^*\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{u} + \int_\Omega\left(p_i\left(\nabla\times\frac{m_i\mathbf{b}}{e\,\|\,\boldsymbol{B}\,\|}\right)\cdot\nabla\boldsymbol{u}\right)\cdot\boldsymbol{u}$$
$$(2) = \int_\Omega \nabla_\perp\left(\frac{m_i p_i}{2e\,\|\,\boldsymbol{B}\,\|}\nabla\cdot(\mathbf{b}\times\boldsymbol{u})\right)\cdot\boldsymbol{u} + \int_\Omega \mathbf{b}\times\nabla\left(\frac{m_i p_i}{2e\,\|\,\boldsymbol{B}\,\|}\nabla_\perp\cdot\boldsymbol{u}\right)\cdot\boldsymbol{u}$$
Firstly by definition of the diamagnetic velocity we have that

$$\begin{aligned}
(1) &= -\int_\Omega\left(\rho\left(\frac{m_i}{e\rho\,\|\,\boldsymbol{B}\,\|}\mathbf{b}\times\nabla p_i\right)\cdot\nabla\boldsymbol{u}\right)\cdot\boldsymbol{u} + \int_\Omega\left(p_i\left(\nabla\times\frac{m_i\mathbf{b}}{e\,\|\,\boldsymbol{B}\,\|}\right)\cdot\nabla\boldsymbol{u}\right)\cdot\boldsymbol{u}\\
&= -\int_\Omega\left(\left(\frac{m_i}{e\,\|\,\boldsymbol{B}\,\|}\mathbf{b}\times\nabla p_i\right)\cdot\nabla\boldsymbol{u}\right)\cdot\boldsymbol{u} + \int_\Omega\left(p_i\left(\nabla\times\frac{m_i\mathbf{b}}{e\,\|\,\boldsymbol{B}\,\|}\right)\cdot\nabla\boldsymbol{u}\right)\cdot\boldsymbol{u}\\
&= \int_\Omega\left(\left(\nabla p_i\times\frac{m_i}{e\,\|\,\boldsymbol{B}\,\|}\mathbf{b}\right)\cdot\nabla\boldsymbol{u}\right)\cdot\boldsymbol{u} + \int_\Omega\left(\left(p_i\nabla\times\frac{m_i\mathbf{b}}{e\,\|\,\boldsymbol{B}\,\|}\right)\cdot\nabla\boldsymbol{u}\right)\cdot\boldsymbol{u}\\
&= \int_\Omega\left(\nabla p_i\times\frac{m_i}{e\,\|\,\boldsymbol{B}\,\|}\mathbf{b}\right)\cdot\nabla\left(\frac{\|\,\boldsymbol{u}\,\|^2}{2}\right) + \int_\Omega\left(p_i\nabla\times\frac{m_i\mathbf{b}}{e\,\|\,\boldsymbol{B}\,\|}\right)\cdot\nabla\left(\frac{\|\,\boldsymbol{u}\,\|^2}{2}\right)\\
&= \int_\Omega\left(\nabla\times\left(\frac{m_i}{e\,\|\,\boldsymbol{B}\,\|}\mathbf{b}p_i\right)\right)\cdot\nabla\left(\frac{\|\,\boldsymbol{u}\,\|^2}{2}\right)\\
&= \int_\Omega\left(\frac{m_i}{e\,\|\,\boldsymbol{B}\,\|}\mathbf{b}p_i\right)\cdot\nabla\times\nabla\left(\frac{\|\,\boldsymbol{u}\,\|^2}{2}\right) = 0
\end{aligned}$$

Now we consider the part of the gyro-viscous tensor

$$\begin{aligned}
(2) &= \int_\Omega \nabla_\perp\left(\frac{m_i p_i}{2e\,\|\,\boldsymbol{B}\,\|}\nabla\cdot(\mathbf{b}\times\boldsymbol{u})\right)\cdot\boldsymbol{u} + \int_\Omega \mathbf{b}\times\nabla\left(\frac{m_i p_i}{2e\,\|\,\boldsymbol{B}\,\|}\nabla_\perp\cdot\boldsymbol{u}\right)\cdot\boldsymbol{u}\\
&= -\int_\Omega\left(\frac{m_i p_i}{2e\,\|\,\boldsymbol{B}\,\|}\nabla\cdot(\mathbf{b}\times\boldsymbol{u})\right)\nabla_\perp\cdot\boldsymbol{u} + \int_\Omega \mathbf{b}\times\nabla\left(\frac{m_i p_i}{2e\,\|\,\boldsymbol{B}\,\|}\nabla_\perp\cdot\boldsymbol{u}\right)\cdot\boldsymbol{u}\\
&= -\int_\Omega\left(\frac{m_i p_i}{2e\,\|\,\boldsymbol{B}\,\|}\nabla\cdot(\mathbf{b}\times\boldsymbol{u})\right)\nabla_\perp\cdot\boldsymbol{u} - \int_\Omega\left(\nabla\left(\frac{m_i p_i}{2e\,\|\,\boldsymbol{B}\,\|}\nabla_\perp\cdot\boldsymbol{u}\right)\times\mathbf{b}\right)\cdot\boldsymbol{u}\\
&= -\int_\Omega\left(\frac{m_i p_i}{2e\,\|\,\boldsymbol{B}\,\|}\nabla\cdot(\mathbf{b}\times\boldsymbol{u})\right)\nabla_\perp\cdot\boldsymbol{u} - \int_\Omega \nabla\left(\frac{m_i p_i}{2e\,\|\,\boldsymbol{B}\,\|}\nabla_\perp\cdot\boldsymbol{u}\right)\cdot(\mathbf{b}\times\boldsymbol{u})\\
&= -\int_\Omega\left(\frac{m_i p_i}{2e\,\|\,\boldsymbol{B}\,\|}\nabla\cdot(\mathbf{b}\times\boldsymbol{u})\right)\nabla_\perp\cdot\boldsymbol{u} - \int_\Omega\left(\frac{m_i p_i}{2e\,\|\,\boldsymbol{B}\,\|}\nabla_\perp\cdot\boldsymbol{u}\right)\nabla\cdot(\mathbf{b}\times\boldsymbol{u}) = 0
\end{aligned}$$

The model conserve the total energy. Consequently we propose to consider this model and begin the reduction for this model

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